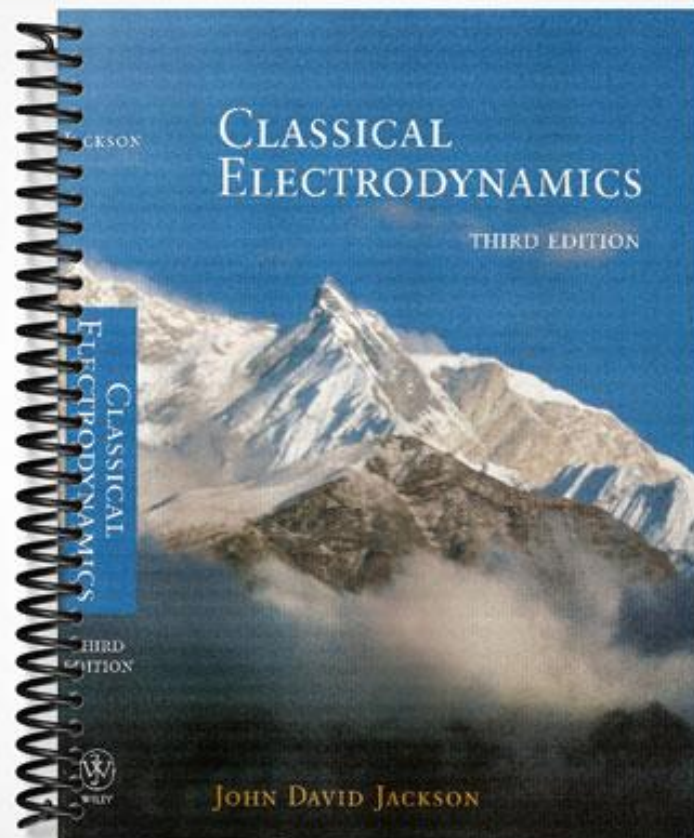


# SOLUTIONS MANUAL



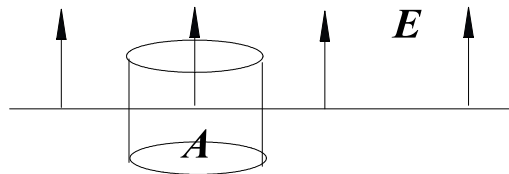
## 1.1

Background:

$$\text{Gauss's Law: } \int \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

Also, a conductor is defined as a material in which the conducting electrons move freely if an external electric field is applied. Thus in static equilibrium, there is no electric field present within a conductor; similarly an electric field parallel to the surface of the conductor would cause charges to move on the surface, and so this electric field cannot exist in static equilibrium. We conclude only electric fields perpendicular to the surface of the conductor can exist.

- a) From the above arguments, the excess charge lies completely on the surface.
- b) Consider a closed hollow conductor. Now bring up a collection of charges on the outside (bring them slowly so that static equilibrium always obtains.) From our above arguments, the electric field lines from the charges never penetrate the conductor, so the hollow region within is shielded. On the other hand, if charges are placed within the hollow part of the conductor, electric fields exist throughout the interior because Gauss's law shows the electric field is non-zero for any surface within the interior which encloses the charges brought in.
- c) Consider the Gaussian pillbox



Gauss's Law gives

$$\int \vec{E} \cdot d\vec{a} = AE = \frac{A\sigma}{\epsilon_0} \rightarrow E = \frac{\sigma}{\epsilon_0}$$

## 1.3

In general we take the charge density to be of the form  $\rho = f(\vec{r})\delta$ , where  $f(\vec{r})$  is determined by physical constraints, such as  $\int \rho d^3x = Q$ .

a) variables:  $r, \theta, \phi$ .  $d^3x = d\phi d\cos\theta r^2 dr$

$$\rho = f(\vec{r})\delta(r - R) = f(r)\delta(r - R) = f(R)\delta(r - R)$$

$$\int \rho d^3x = f(R) \int r^2 dr d\Omega d(r - R) = 4\pi f(R)R^2 = Q \rightarrow f(R) = \frac{Q}{4\pi R^2}$$


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$$\rho(\vec{r}) = \frac{Q}{4\pi R^2} \delta(r - R)$$

b) variables:  $r, \phi, z$ .  $d^3x = d\phi dz r dr$

$$\rho(\vec{r}) = f(\vec{r})\delta(r - b) = f(b)\delta(r - b)$$

$$\int \rho d^3x = f(b) \int dz d\phi r dr \delta(r - b) = 2\pi f(b)bL = \lambda L \rightarrow f(b) = \frac{\lambda}{2\pi b}$$

$$\rho(\vec{r}) = \frac{\lambda}{2\pi b} \delta(r - b)$$

c) variables:  $r, \phi, z$ .  $d^3x = d\phi dz r dr$  Choose the center of the disk at the origin, and the z-axis perpendicular to the plane of the disk

$$\rho(\vec{r}) = f(\vec{r})\delta(z)\theta(R - r) = f\delta(z)\theta(R - r)$$

where  $\theta(R - r)$  is a step function.

$$\int \rho d^3x = f \int \delta(z)\theta(R - r) d\phi dz r dr = 2\pi \frac{R^2}{2} f = Q \rightarrow f = \frac{Q}{\pi R^2}$$

$$\rho(\vec{r}) = \frac{Q}{\pi R^2} \delta(z)\theta(R - r)$$

d) variables:  $r, \theta, \phi$ .  $d^3x = d\phi d\cos\theta r^2 dr$

$$\rho(\vec{r}) = f(\vec{r})\delta(\cos\theta)\theta(R - r) = f(r)\delta(\cos\theta)\theta(R - r)$$

$$\int \rho d^3x = \int f(r)\delta(\cos\theta)\theta(R - r) d\phi d\cos\theta r^2 dr = \int_0^R [f(r)r] r dr d\phi = 2\pi N \int_0^R r dr = \pi R^2 N = Q$$

where I've used the fact that  $r dr d\phi$  is an element of area and that the charge density is uniformly distributed over area.

$$\rho(\vec{r}) = \frac{Q}{\pi R^2 r} \delta(\cos\theta)\theta(R - r)$$

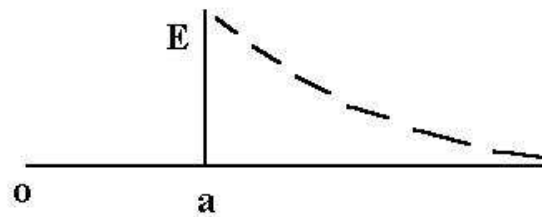
Gauss's Law:

$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

a) Conducting sphere: all of the charge is on the surface  $\sigma = \frac{Q}{4\pi a^2}$

$$E4\pi r^2 = 0, \quad r < a \quad E4\pi r^2 = \frac{Q}{\epsilon_0}, \quad r \geq a$$

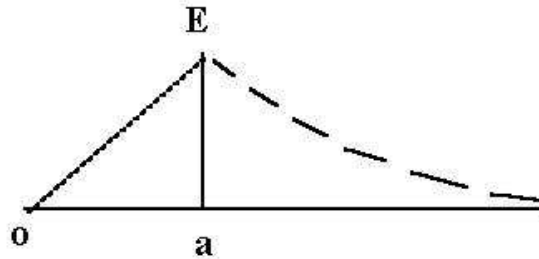
$$E = 0, \quad r < a \quad \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}, \quad r \geq a$$



b) Uniform charge density:  $\rho = \frac{Q}{\frac{4}{3}\pi a^3}$ ,  $r < a$ ,  $\rho = 0$ ,  $r \geq a$ .

$$E4\pi r^2 = \frac{Qr^3}{\epsilon_0 a^3} \rightarrow E = \frac{Qr}{4\pi\epsilon_0 a^3}, \quad r < a$$

$$E4\pi r^2 = \frac{Q}{\epsilon_0} \rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r \geq a$$



c)  $\rho = A r^n$

$$Q = 4\pi \int r^2 dr A r^n = 4\pi A a^{n+3}/(n+3) \rightarrow \rho = \frac{(n+3)Q}{4\pi a^{n+3}} r^n, \quad r < a$$

$$\rho = 0, \quad r \geq a$$

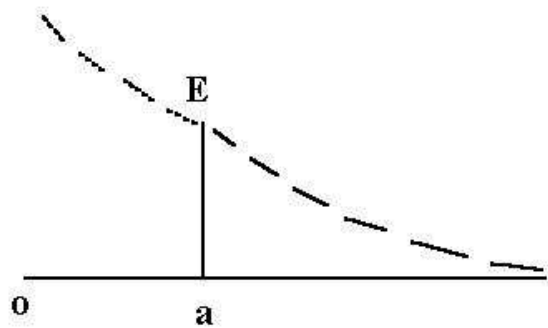
$$E4\pi r^2 = \frac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}} 4\pi r^{n+3}/(n+3) \rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2} \left( \frac{r^{n+3}}{a^{n+3}} \right), \quad r < a$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r \geq a$$

1.  $n = -2$ .

$$E = \frac{Q}{4\pi\epsilon_0 r a}, \quad r < a$$

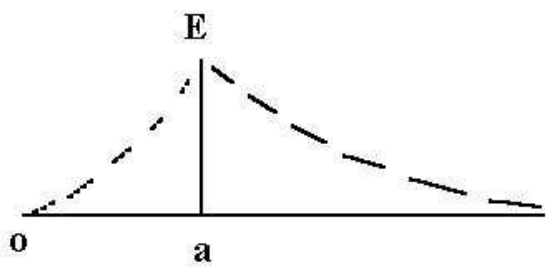
$$E = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r \geq a$$



2.  $n = 2$ .

$$E = \frac{Qr^3}{4\pi\epsilon_0 a^5}, \quad r < a$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r \geq a$$



1.5

$$\phi(\vec{r}) = \frac{qe^{-ar}\left(1 + \frac{ar}{2}\right)}{4\pi\epsilon_0 r}$$

$$\nabla^2\phi(\vec{r}) = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right)$$

$$\nabla^2\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} \left( \frac{e^{-ar}}{r} + \frac{\alpha}{2} e^{-ar} \right) \right]$$

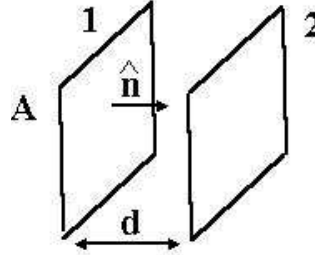
Using  $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$

$$\begin{aligned} \nabla^2\phi &= \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left[ -\alpha r e^{-ar} + e^{-ar} r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \right) - \frac{\alpha^2}{2} r^2 e^{-ar} \right] \\ &= \frac{q}{4\pi\epsilon_0} \left[ -\frac{\alpha}{r^2} e^{-ar} + \frac{\alpha^2 e^{-ar}}{r} + \frac{\alpha e^{-ar}}{r^2} - 4\pi\delta(\vec{r}) - \frac{\alpha^2 e^{-ar}}{r} + \frac{\alpha^3 e^{-ar}}{2} \right] \\ &= -\frac{1}{\epsilon_0} \left[ q\delta(\vec{r}) - \frac{q\alpha^3}{8\pi} e^{-ar} \right] \\ \rho(\vec{r}) &= q\delta(\vec{r}) - \frac{q\alpha^3}{8\pi} e^{-ar} \end{aligned}$$

That is, the charge distribution consists of a positive point charge at the origin, plus an exponentially decreasing negatively charged cloud.

We will be using Gauss's law  $\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

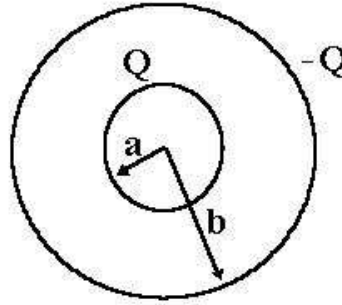
a) 1) Parallel plate capacitor



From Gauss's law  $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0} = \frac{\phi_{12}}{d} \rightarrow Q = \frac{A\epsilon_0\phi_{12}}{d}$

$$W = \frac{\epsilon_0}{2} \int E^2 d^3x = \frac{\epsilon_0 E^2 A d}{2} = \frac{\epsilon_0 \left( \frac{Q}{A\epsilon_0} \right)^2 A d}{2} = \frac{1}{2\epsilon_0} \frac{Q^2}{A} d = \frac{1}{2\epsilon_0} \frac{\left( \frac{A\epsilon_0\phi_{12}}{d} \right)^2}{A} d = \frac{1}{2} \epsilon_0 A \frac{\phi_{12}^2}{d}$$

2) Spherical capacitor



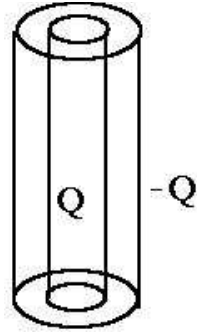
From Gauss's law,  $E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$ ,  $a < r < b$ .

$$\phi_{12} = \int_a^b E dr = \frac{Q}{4\pi\epsilon_0} \int_a^b r^{-2} dr = \frac{1}{4} \frac{Q}{\pi\epsilon_0} \frac{(b-a)}{ba} \rightarrow Q = \frac{4\pi\epsilon_0 ba \phi_{12}}{(b-a)}$$

$$W = \frac{\epsilon_0}{2} \int E^2 d^3x = \frac{\epsilon_0}{2} \left( \frac{Q}{4\pi\epsilon_0} \right)^2 \int_a^b 4\pi \frac{r^2 dr}{r^4} = \frac{\epsilon_0}{2} \left( \frac{Q}{4\pi\epsilon_0} \right)^2 \left( -4\pi \frac{b-a}{ba} \right) = \frac{1}{8\epsilon_0} \frac{Q^2}{\pi} \frac{(b-a)}{ba}$$

$$W = \frac{1}{8\epsilon_0} \frac{\left( \frac{4\pi\epsilon_0 ba \phi_{12}}{(b-a)} \right)^2}{\pi} \frac{(b-a)}{ba} = 2\pi\epsilon_0 ba \frac{\phi_{12}^2}{(b-a)}$$

3) Cylindrical capacitor



From Gauss's law,  $E2\pi rL = \frac{\lambda L}{\epsilon_0} = \frac{Q}{\epsilon_0}$

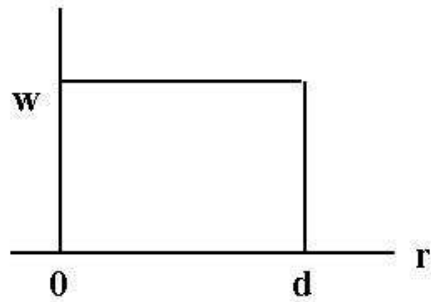
$$\phi_{12} = \int_a^b E dr = \frac{Q}{2\pi\epsilon_0 L} \int_a^b \frac{dr}{r} = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

$$W = \frac{\epsilon_0}{2} \int E^2 d^3x = \frac{\epsilon_0}{2} \left( \frac{Q}{2\pi\epsilon_0 L} \right)^2 2\pi L \int_a^b \frac{r dr}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{Q^2}{L} \ln\left(\frac{b}{a}\right)$$

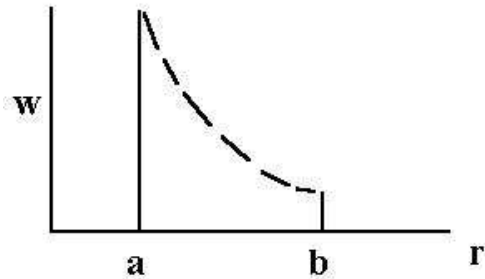
$$W = \frac{1}{4\pi\epsilon_0} \frac{\left( \frac{2\pi\epsilon_0 L \phi_{12}}{\ln\left(\frac{b}{a}\right)} \right)^2}{L} \ln\left(\frac{b}{a}\right) = \pi\epsilon_0 L \frac{\phi_{12}^2}{\ln\frac{b}{a}}$$

b)  $w = \frac{\epsilon_0}{2} E^2$

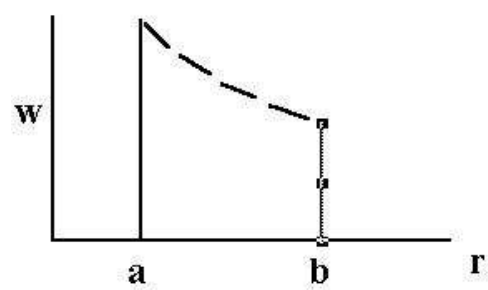
1)  $w(r) = \frac{\epsilon_0}{2} \left( \frac{Q}{A\epsilon_0} \right)^2 = \frac{1}{2\epsilon_0} \frac{Q^2}{A^2} \quad 0 < r < d, = 0 \text{ otherwise.}$



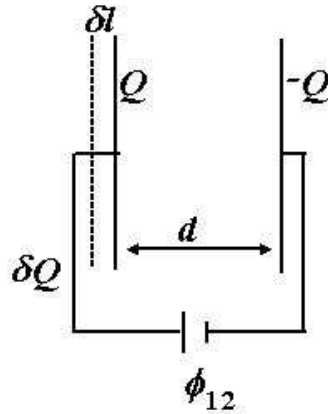
2)  $w(r) = \frac{\epsilon_0}{2} \left( \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \right)^2 = \frac{1}{32\epsilon_0\pi^2} \frac{Q^2}{r^4}, \quad a < r < b, = 0, \text{ otherwise}$



3)  $w(r) = \frac{\epsilon_0}{2} \left( \frac{Q}{2\pi\epsilon_0 L r} \right)^2 = \frac{1}{8\epsilon_0} \frac{Q^2}{\pi^2 L^2 r^2}, \quad = 0 \text{ otherwise.}$



I will be using the principle of virtual work. In the figure below,  $F\delta l$  is the work done by an external force. If  $F$  is along  $\delta l$  (ie. is positive), then the force between the plates is attractive. This work goes into increasing the electrostatic energy carried by the electric field and into forcing charge into the battery holding the plates at constant potential  $\phi_{12}$ .



Conservation of energy gives

$$F\delta l = \delta W + \delta Q\phi_{12}$$

or

$$F = \frac{\partial W}{\partial l} + \left| \frac{\partial Q}{\partial l} \right| \phi_{12}$$

From problem 1.8,

a) Charge fixed.

1) Parallel plate capacitor

$$W = \frac{1}{2} \epsilon_0 A \frac{\phi_{12}^2}{d}, \phi_{12} = \frac{dQ}{A\epsilon_0} \rightarrow W = \frac{1}{2} \epsilon_0 A \frac{\left(\frac{dQ}{A\epsilon_0}\right)^2}{d} = \frac{d}{2\epsilon_0 A} Q^2$$

$$\frac{\partial Q}{\partial l} = 0, F = \frac{\partial W}{\partial l} = \frac{Q^2}{2\epsilon_0 A} \text{ (attractive)}$$

2) Parallel cylinder capacitor

$$\phi_{12} = \frac{\lambda}{\epsilon_0} \ln\left(\frac{d}{a}\right), a = \sqrt{a_1 a_2}$$

$$W = \frac{1}{2} Q \phi_{12} \rightarrow F = \frac{\partial W}{\partial l} = \frac{1}{2} Q \frac{\lambda}{\epsilon_0} \frac{\partial}{\partial d} \ln\left(\frac{d}{a}\right) = \frac{1}{2} \frac{\lambda Q}{\epsilon_0 d} \text{ (attractive)}$$

b) Potential fixed

1) Parallel plate capacitor

$$\text{Using Gauss's law, } Q = \frac{\phi_{12} A \epsilon_0}{d}, \left| \frac{\partial}{\partial l} Q \right| = \frac{\phi_{12} A \epsilon_0}{d^2}$$

$$\rightarrow F = -\frac{1}{2}\epsilon_0 A \frac{\phi_{12}^2}{d^2} + \frac{\phi_{12}^2 A \epsilon_0}{d^2} = \frac{1}{2}\epsilon_0 A \frac{\phi_{12}^2}{d^2} = \frac{1}{2}\epsilon_0 A \frac{\left(\frac{Qd}{\epsilon_0 A}\right)^2}{d^2} = \frac{1}{2\epsilon_0 A} Q^2$$

2) Parallel cylinder capacitor

$$W = \frac{1}{2} Q \phi_{12}, \text{ and } Q = \frac{\epsilon_0 L \phi_{12}}{\ln(\frac{d}{a})}$$

so

$$W = \frac{1}{2} \frac{\epsilon_0 L \phi_{12}^2}{\ln(\frac{d}{a})}, \quad \frac{\partial W}{\partial l} = -\frac{1}{2} \epsilon_0 L \frac{\phi_{12}^2}{(\ln^2 \frac{d}{a}) d}$$

$$\left| \frac{\partial Q}{\partial l} \right| = \epsilon_0 L \frac{\phi_{12}}{(\ln^2 \frac{d}{a}) d}$$

$$F = -\frac{1}{2} \epsilon_0 L \frac{\phi_{12}^2}{(\ln^2 \frac{d}{a}) d} + \epsilon_0 L \frac{\phi_{12}^2}{(\ln^2 \frac{d}{a}) d} = \frac{1}{2} \epsilon_0 L \frac{\phi_{12}^2}{(\ln^2 \frac{d}{a}) d}$$

1.10 I will base the solution on the application of Green's theorem, which results in eq. 1.36 from the textbook:

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}')}{R} d^3x' + \frac{1}{4\pi} \oint_S \left[ \frac{1}{R} \frac{\partial\phi}{\partial n'} - \phi \frac{\partial}{\partial n'} \left( \frac{1}{R} \right) \right] da'$$

Since the volume includes no charge, the first term on the rhs vanishes. For the second term

$$\frac{\partial\phi}{\partial n'} = \vec{\nabla}\phi \cdot \hat{n}' = -\vec{E} \cdot \hat{n}'$$

Note

$$\oint_S \vec{E} \cdot \hat{n}' da' = \int_V \vec{\nabla}' \cdot \vec{E} d^3x' \text{ by the divergence theorem}$$

Using the fact that

$$\vec{\nabla}' \cdot \vec{E} = \rho(\vec{x}')/\epsilon_0$$

then the second term of the first equation also vanishes, since the volume integrated over contains no charge. Since  $\frac{\partial}{\partial n'} \left( \frac{1}{R} \right) = -\frac{1}{R^2}$ , where  $R$  is the radius of the sphere, and I'm taking the origin at the center of the sphere,

$$\phi(\vec{x}) = \frac{1}{4\pi R^2} \oint_S \phi(\vec{x}') da' = \text{mean value of the potential over the sphere.}$$

## More Problems for Chapter 1

### **Problem 1.5**

From Poisson's equation  $\nabla^2\Phi = -\rho/\epsilon_0$ , we have the charge density  $\rho = -\epsilon_0\nabla^2\Phi$ . Let  $f \equiv \frac{1}{r}$  and  $g \equiv e^{-\alpha r}(1 + \frac{\alpha r}{2})$ , then

$$\rho = -\epsilon_0\nabla^2\Phi = -\frac{q}{4\pi}(g\nabla^2f + 2\nabla f \cdot \nabla g + f\nabla^2g)$$

Note that

$$\nabla f = -\frac{\vec{r}}{r^3}, \quad \nabla^2 f = -4\pi\delta^3(\vec{r})$$

$$\nabla g = -\frac{1}{2}\alpha e^{-\alpha r}(1 + \alpha r)\frac{\vec{r}}{r}, \quad \nabla^2 g = \frac{\alpha}{2r}e^{-\alpha r}(-2 - 2\alpha r + \alpha^2 r^2)$$

Plug them into the charge density

$$\rho = -\frac{q}{4\pi}(-4\pi\delta^3(\vec{r}) + \frac{1}{2}\alpha^3 e^{-\alpha r}) = q\delta^3(\vec{r}) - \frac{q}{8\pi}\alpha^3 e^{-\alpha r}$$

The first term represents a point charge  $q$  at the origin and the second term is due to a continuous distributed volume charge. Note that the total charge is zero:

$$\int \rho d^3x = 0$$

### **Problem 1.6**

(a) Applying Gauss's law to the plate with a Gaussian pillbox, one gets the electric field due to a flat surface charge distribution to be  $\sigma/2\epsilon_0$ , where  $\sigma = Q/A$  is the surface charge density. The contributions from the two plates add up in between the two plates and cancel outside. The total electric field in between the two plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}.$$

The potential difference between the two plates

$$V \equiv \Phi_+ - \Phi_- = -\int_-^+ \vec{E} \cdot d\vec{\ell} = \frac{Qd}{\epsilon_0 A}$$

The capacitance

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}.$$

(b) Assuming the inner sphere has charge  $Q$  and the outer has  $-Q$ , applying Gauss's law with a Gaussian sphere of radius  $r$  ( $a < r < b$ ):

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

The potential difference

$$V \equiv \Phi_+ - \Phi_- = -\int_-^+ \vec{E} \cdot d\vec{\ell} = \frac{Q}{4\pi\epsilon_0} \frac{b-a}{ab}$$

The capacitance is therefore

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

(c) Again assuming the inner cylinder has charge  $Q$  and the outer has  $-Q$ , applying Gauss's law with a cylindrical surface as the Gaussian surface:

$$\vec{E} = \frac{Q}{2\pi\epsilon_0} \frac{\vec{r}}{r^2}$$

The potential difference

$$V = - \int_{-}^{+} \vec{E} \cdot d\vec{\ell} = \frac{Q}{2\pi\epsilon_0} \ln \frac{b}{a}$$

The capacitance per unit length

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln \frac{b}{a}}$$

### **Problem 1.7**

From Gauss's law, the field due to one conductor is

$$\vec{E} = \frac{Q}{2\pi\epsilon_0} \frac{1}{r}$$

where  $Q$  is the charge per unit length and  $r$  is the perpendicular distance from the point of interest to the conductor. Along the perpendicular line joining the two conductors, the fields due to the two conductors are in the same direction. Therefore, the total field along the line is:

$$E = \frac{Q}{2\pi\epsilon_0} \frac{1}{r_+} + \frac{Q}{2\pi\epsilon_0} \frac{1}{r_-}$$

where  $r_+$  and  $r_-$  are the perpendicular distances to the positively and negatively charged conductors respectively.  $\vec{E}$  points from  $+Q$  to  $-Q$ . The potential difference

$$V = \Phi_+ - \Phi_- = - \int_{-}^{+} \vec{E} \cdot d\vec{\ell} = \frac{Q}{2\pi\epsilon_0} \left\{ \int_{a_1}^{d-a_2} \frac{1}{r_+} dr_+ + \int_{a_2}^{d-a_1} \frac{1}{r_-} dr_- \right\} = \frac{Q}{2\pi\epsilon_0} \ln \frac{(d-a_1)(d-a_2)}{a_1 a_2}$$

The capacitance per unit length

$$C = \frac{Q}{V} = 2\pi\epsilon_0 / \ln \frac{(d-a_1)(d-a_2)}{a_1 a_2} \approx 2\pi\epsilon_0 / \ln \frac{d^2}{(\sqrt{a_1 a_2})^2} = \pi\epsilon_0 / \ln \frac{d}{a}$$

where  $a$  is the geometrical mean of  $a_1$  and  $a_2$ :  $a = \sqrt{a_1 a_2}$ .

### **Problem 1.9**

(a) *Parallel plate capacitor*

The negatively charged plate experiences a field of

$$E = \frac{\sigma}{2\epsilon_0} = \frac{Q}{2\epsilon_0 A}$$

due to the positively charged plate, where  $Q$  is the total charge on the plate. Therefore, the attractive force between the two plates is

$$F = QE = \frac{Q^2}{2\epsilon_0 A}$$

*Parallel cylinder capacitor*

Again, one conductor experiences an electric field of

$$E \approx \frac{Q}{2\pi\epsilon_0} \frac{1}{d}$$

from the other conductor. Here  $Q$  is the charge per unit length. Therefore, the attractive force per unit length between them is

$$F = QE = \frac{Q^2}{2\pi\epsilon_0} \frac{1}{d}$$

(b) The force should be the same except that  $Q$  should be replaced by  $CV$ .

*Parallel plate capacitor*

$$F = \frac{\epsilon_0 AV^2}{2d^2}$$

*Parallel cylinder capacitor*

$$F = \frac{\pi\epsilon_0 V^2}{2d \ln(b/a)^2}$$

## 2.1

We will work in cylindrical coordinate,  $(\rho, z, \phi)$ , with the charge  $q$  located at the point  $\vec{d} = d\hat{z}$ , and the conducting plane is in the  $z = 0$  plane.

Then we know from class the potential is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{|\vec{x} - \vec{d}|} - \frac{q}{|\vec{x} + \vec{d}|} \right]$$

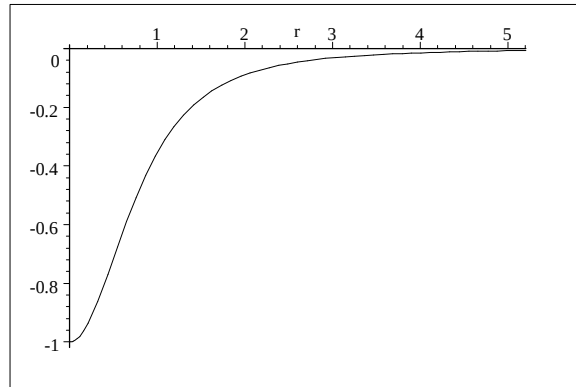
$$E_z = -\frac{q}{4\pi\epsilon_0} \frac{\partial}{\partial z} \left[ \frac{1}{((z-d)^2 + \rho^2)^{1/2}} - \frac{1}{((z+d)^2 + \rho^2)^{1/2}} \right]$$

$$E_z = \frac{q}{4\pi\epsilon_0} \left[ \frac{z-d}{((z-d)^2 + \rho^2)^{3/2}} - \frac{z+d}{((z+d)^2 + \rho^2)^{3/2}} \right]$$

a)  $\sigma = \epsilon_0 E_z(z = 0)$

$$\sigma = \epsilon_0 \frac{q}{4\pi\epsilon_0} \left[ \frac{-d}{((-d)^2 + \rho^2)^{3/2}} - \frac{+d}{((+d)^2 + \rho^2)^{3/2}} \right] = -\frac{q}{2\pi d^2} \frac{1}{\left(1^2 + \left(\frac{\rho}{d}\right)^2\right)^{\frac{3}{2}}}$$

Plotting  $\frac{-1}{(1^2 + r^2)^{\frac{3}{2}}}$  gives



b) Force of charge on plane

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q(-q)}{(2d)^2} (-\hat{z}) = \frac{1}{4 \times 4\pi\epsilon_0} \frac{q^2}{d^2} \hat{z}$$

c)

$$\frac{F}{A} = w = \frac{\epsilon_0}{2} E^2 = \frac{\sigma^2}{2\epsilon_0} = \frac{1}{2\epsilon_0} \left( -\frac{q}{2\pi d^2} \frac{1}{\left(1^2 + \left(\frac{\rho}{d}\right)^2\right)^{\frac{3}{2}}} \right)^2 = \frac{1}{8\epsilon_0} \frac{q^2}{\pi^2 d^4 \left(1 + \frac{\rho^2}{d^2}\right)^3}$$

$$F = 2\pi \frac{q^2}{8\epsilon_0 \pi^2 d^4} \int_0^\beta \frac{\rho}{\left(1 + \frac{\rho^2}{d^2}\right)^3} d\rho = 2\pi \frac{q^2}{8\epsilon_0 \pi^2 d^4} \left( \frac{1}{4} d^2 \right) = \frac{1}{4 \times 4\pi\epsilon_0} \frac{q^2}{d^2}$$

d)

$$W = \int_d^\beta F dz = \frac{q^2}{4 \times 4\pi\epsilon_0} \int_d^\beta \frac{dz}{z^2} = \frac{q^2}{4 \times 4\pi\epsilon_0 d}$$

e)

$$W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i,j,i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = -\frac{q^2}{2 \times 4\pi\epsilon_0 d}$$

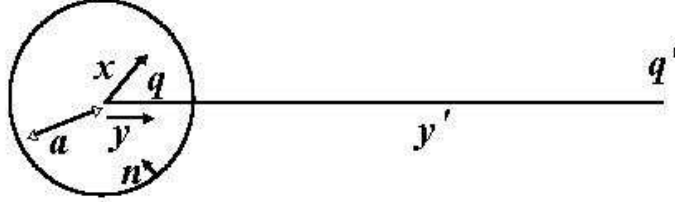
Notice parts d) and e) are not equal in magnitude, because in d) the image moves when  $q$  moves.

f) 1 Angstrom =  $10^{-10}\text{m}$ ,  $q = e = 1.6 \times 10^{-19}\text{C}$ .

$$W = \frac{q^2}{4 \times 4\pi\epsilon_0 d} = e \frac{e}{4 \times 4\pi\epsilon_0 d} = e \frac{1.6 \times 10^{-19}}{4 \times 10^{-10}} 9 \times 10^9 \text{ V} = 3.6 \text{ eV}$$

## 2.2

The system is described by



a) Using the method of images

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}'|} \right]$$

with  $y' = \frac{a^2}{y}$ , and  $q' = -q \frac{a}{y}$

b)  $\sigma = -\epsilon_0 \frac{\partial \phi}{\partial n} \Big|_{x=a} = +\epsilon_0 \frac{\partial \phi}{\partial x} \Big|_{x=a}$

$$\sigma = \epsilon_0 \frac{1}{4\pi\epsilon_0} \frac{\partial}{\partial x} \left[ \frac{q}{(x^2 + y^2 - 2xy \cos \gamma)^{1/2}} + \frac{q'}{(x^2 + y'^2 - 2xy' \cos \gamma)^{1/2}} \right]$$

$$\sigma = -q \frac{1}{4\pi} \frac{a \left( 1 - \frac{y^2}{a^2} \right)}{(y^2 + a^2 - 2ay \cos \gamma)^{3/2}}$$

Note

$$q_{induced} = a^2 \int \sigma d\Omega = -q \frac{1}{4\pi} a^2 2\pi a \left( 1 - \frac{y^2}{a^2} \right) \int_{-1}^1 \frac{dx}{(y^2 + a^2 - 2ayx)^{3/2}}, \text{ where } x = \cos \gamma$$

$$q_{induced} = -\frac{q}{2} a(a^2 - y^2) \frac{2}{a(a^2 - y^2)} = -q$$

c)

$$|F| = \left| \frac{qq'}{4\pi\epsilon_0(y' - y)^2} \right| = \frac{1}{4\pi\epsilon_0} \frac{q^2 ay}{(a^2 - y^2)}, \text{ the force is attractive, to the right.}$$

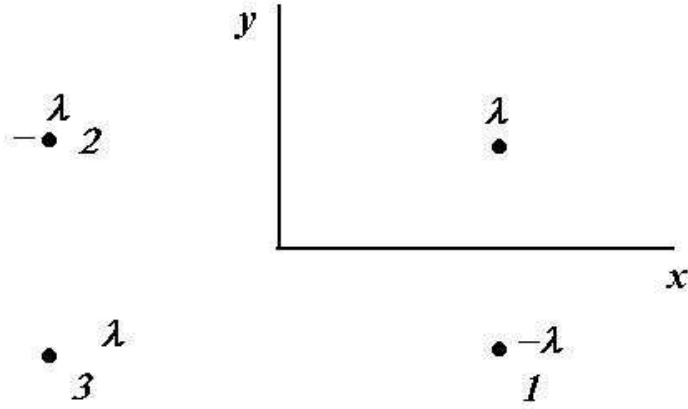
d) If the conductor were fixed at a different potential, or equivalently if extra charge were put on the conductor, then the potential would be

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}'|} \right] + V$$

and obviously the electric field in the sphere and induced charge on the inside of the sphere would remain unchanged.

## 2.3

The system is described by



a) Given the potential for a line charge in the problem, we write down the solution from the figure,

$$\phi_T = \frac{\lambda}{4\pi\epsilon_0} \left[ \ln \frac{R^2}{(\vec{x} - \vec{x}_o)^2} - \ln \frac{R^2}{(\vec{x} - \vec{x}_{o1})^2} - \ln \frac{R^2}{(\vec{x} - \vec{x}_{o2})^2} + \ln \frac{R^2}{(\vec{x} - \vec{x}_{o3})^2} \right]$$

Looking at the figure when  $y = 0$ ,  $(\vec{x} - \vec{x}_o)^2 = (\vec{x} - \vec{x}_{o1})^2$ ,  $(\vec{x} - \vec{x}_{o2})^2 = (\vec{x} - \vec{x}_{o3})^2$ , so  $\phi_T|_{y=0} = 0$   
 Similarly, when  $x = 0$ ,  $(\vec{x} - \vec{x}_o)^2 = (\vec{x} - \vec{x}_{o2})^2$ ,  $(\vec{x} - \vec{x}_{o1})^2 = (\vec{x} - \vec{x}_{o3})^2$ , so  $\phi_T|_{x=0} = 0$   
 On the surface  $\phi_T = 0$ , so  $\delta\phi_T = 0$ , however,

$$\delta\phi_T = \frac{\partial\phi_T}{\partial x_t} \delta x_t = 0 \rightarrow \frac{\partial\phi_T}{\partial x_t} = 0 \rightarrow E_t = 0$$

b) We remember

$$\sigma = -\epsilon_0 \frac{\partial\phi_T}{\partial y} = \frac{-\lambda}{\pi} \left[ \frac{y_0}{(x - x_0)^2 + y_0^2} - \frac{y_0}{(x + x_0)^2 + y_0^2} \right]$$

where I've applied the symmetries derived in a). Let

$$\sigma/\lambda = \frac{-1}{\pi} \left[ \frac{y_0}{(x - x_0)^2 + y_0^2} - \frac{y_0}{(x + x_0)^2 + y_0^2} \right]$$

This is an easy function to plot for various combinations of the position of the original line charge  $(x_0, y_0)$ .

c) If we integrate over a strip of width  $\Delta z$ , we find, where we use the integral

$$\int_0^\infty \frac{1}{(x \mp x_0)^2 + y_0^2} dx = \frac{1}{2} \frac{\pi \pm 2 \arctan \frac{x_0}{y_0}}{y_0}$$

$$\Delta Q = \int_0^\infty \sigma dx \Delta z \rightarrow \frac{\Delta Q}{\Delta z} = \int_0^\infty \sigma dx = \frac{-2\lambda}{\pi} \tan^{-1} \left( \frac{x_0}{y_0} \right)$$

and the total charge induced on the plane is  $-\infty$ , as expected.

d)

Expanding

$$\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right) - \ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right) - \ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right) + \ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right)$$

to lowest non-vanishing order in  $x_0, y_0$  gives

$$16 \frac{xy}{(x^2+y^2)^2} y_0 x_0$$

so

$$\phi \rightarrow \phi_{asym} = \frac{4\lambda}{\pi\epsilon_0} \frac{xy}{(x^2+y^2)^2} y_0 x_0$$

This is the quadrupole contribution.

2.5

a)

$$W = \int_r^\infty |F| dy = \frac{q^2 a}{4\pi\epsilon_0} \int_r^\infty \frac{dy}{y^3 \left(1 - \frac{a^2}{y^2}\right)^2} = \frac{q^2 a}{8\pi\epsilon_0 (r^2 - a^2)}$$

Let us compare this to disassemble the charges

$$-W' = -\frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{4\pi\epsilon_0} \left[ \frac{aq^2}{r} \frac{1}{r \left(1 - \frac{a^2}{r^2}\right)} \right] = \frac{q^2 a}{4\pi\epsilon_0 (r^2 - a^2)} > W$$

The reason for this difference is that in the first expression  $W$ , the image charge is moving and changing size, whereas in the second, they don't.

b) In this case

$$W = \int_r^\infty |F| dy = \frac{q}{4\pi\epsilon_0} \left[ \int_r^\infty \frac{Q dy}{y^2} - qa^3 \int_r^\infty \frac{(2y^2 - a^2)}{y(y^2 - a^2)^2} dy \right]$$

Using standard integrals, this gives

$$W = \frac{1}{4\pi\epsilon_0} \left[ \frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r} \right]$$

On the other hand

$$-W' = \frac{1}{4\pi\epsilon_0} \left[ \frac{aq^2}{(r^2 - a^2)} - \frac{q(Q + \frac{a}{r}q)}{r} \right] = \frac{1}{4\pi\epsilon_0} \left[ \frac{aq^2}{(r^2 - a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r} \right]$$

The first two terms are larger than those found in  $W$  for the same reason as found in a), whereas the last term is the same, because  $Q$  is fixed on the sphere.