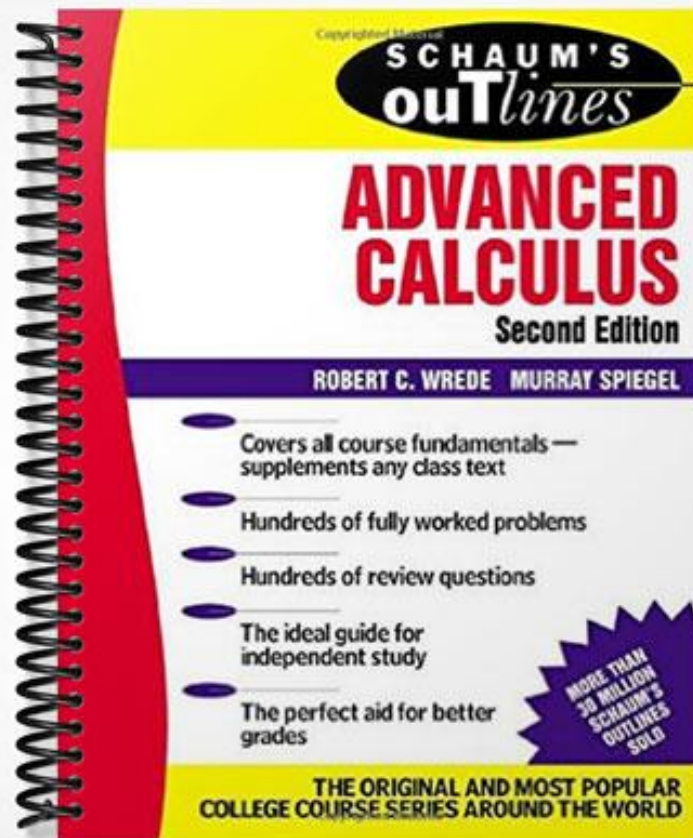
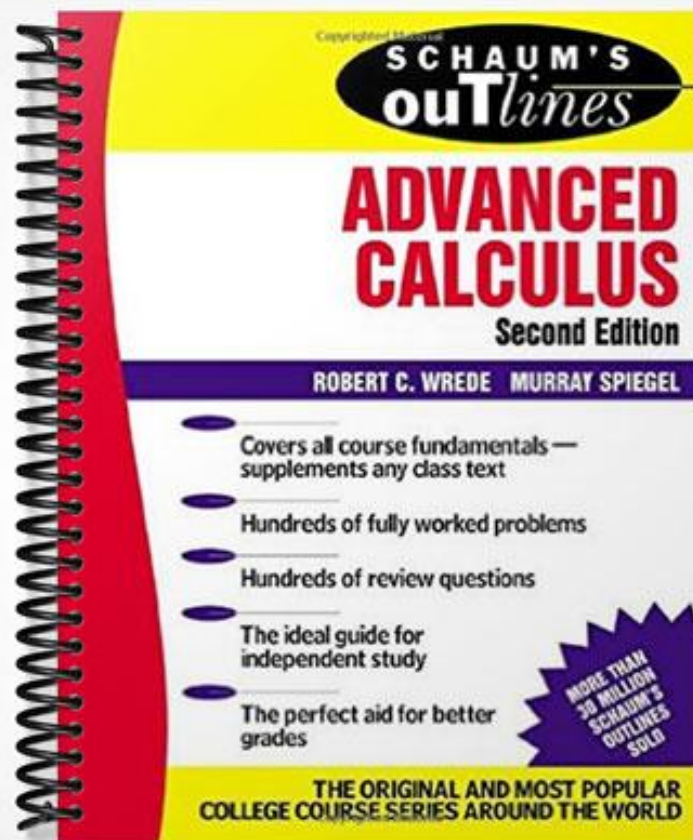


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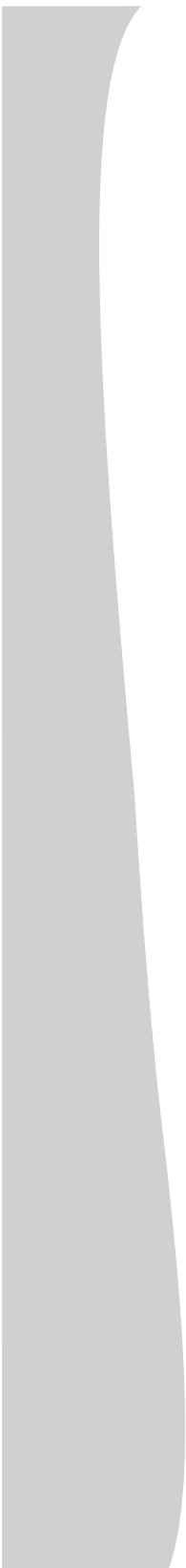
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Second Edition

ROBERT WREDE, Ph.D.

MURRAY R. SPIEGEL, Ph.D.

*Former Professor and Chairman of Mathematics
Rensselaer Polytechnic Institute
Hartford Graduate Center*

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PREFACE

A key ingredient in learning mathematics is problem solving. This is the strength, and no doubt the reason for the longevity of Professor Spiegel's advanced calculus. His collection of solved and unsolved problems remains a part of this second edition.

Advanced calculus is not a single theory. However, the various sub-theories, including vector analysis, infinite series, and special functions, have in common a dependency on the fundamental notions of the calculus. An important objective of this second edition has been to modernize terminology and concepts, so that the interrelationships become clearer. For example, in keeping with present usage functions of a real variable are automatically single valued; differentials are defined as linear functions, and the universal character of vector notation and theory are given greater emphasis. Further explanations have been included and, on occasion, the appropriate terminology to support them.

The order of chapters is modestly rearranged to provide what may be a more logical structure.

A brief introduction is provided for most chapters. Occasionally, a historical note is included; however, for the most part the purpose of the introductions is to orient the reader to the content of the chapters.

I thank the staff of McGraw-Hill. Former editor, Glenn Mott, suggested that I take on the project. Peter McCurdy guided me in the process. Barbara Gilson, Jennifer Chong, and Elizabeth Shannon made valuable contributions to the finished product. Joanne Slike and Maureen Walker accomplished the very difficult task of combining the old with the new and, in the process, corrected my errors. The reviewer, Glenn Ledder, was especially helpful in the choice of material and with comments on various topics.

ROBERT C. WREDE

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CHAPTER 1

Numbers

Mathematics has its own language with numbers as the alphabet. The language is given structure with the aid of connective symbols, rules of operation, and a rigorous mode of thought (logic). These concepts, which previously were explored in elementary mathematics courses such as geometry, algebra, and calculus, are reviewed in the following paragraphs.

SETS

Fundamental in mathematics is the concept of a *set*, *class*, or *collection* of objects having specified characteristics. For example, we speak of the set of all university professors, the set of all letters $A, B, C, D, \dots, Z$ of the English alphabet, and so on. The individual objects of the set are called *members* or *elements*. Any part of a set is called a *subset* of the given set, e.g., A, B, C is a subset of $A, B, C, D, \dots, Z$. The set consisting of no elements is called the *empty set* or *null set*.

REAL NUMBERS

The following types of numbers are already familiar to the student:

1. **Natural numbers** $1, 2, 3, 4, \dots$, also called *positive integers*, are used in counting members of a set. The symbols varied with the times, e.g., the Romans used I, II, III, IV, $\dots$. The *sum* $a + b$ and *product* $a \cdot b$ or ab of any two natural numbers a and b is also a natural number. This is often expressed by saying that the set of natural numbers is *closed* under the operations of *addition* and *multiplication*, or satisfies the *closure property* with respect to these operations.
2. **Negative integers and zero** denoted by $-1, -2, -3, \dots$ and 0 , respectively, arose to permit solutions of equations such as $x + b = a$, where a and b are any natural numbers. This leads to the operation of *subtraction*, or *inverse of addition*, and we write $x = a - b$.

The set of positive and negative integers and zero is called the set of *integers*.

3. **Rational numbers** or *fractions* such as $\frac{2}{3}, -\frac{5}{4}, \dots$ arose to permit solutions of equations such as $bx = a$ for all integers a and b , where $b \neq 0$. This leads to the operation of *division*, or *inverse of multiplication*, and we write $x = a/b$ or $a \div b$ where a is the *numerator* and b the *denominator*.

The set of integers is a subset of the rational numbers, since integers correspond to rational numbers where $b = 1$.

4. **Irrational numbers** such as $\sqrt{2}$ and π are numbers which are not rational, i.e., they cannot be expressed as a/b (called the *quotient* of a and b), where a and b are integers and $b \neq 0$.

The set of rational and irrational numbers is called the set of *real numbers*.

DECIMAL REPRESENTATION OF REAL NUMBERS

Any real number can be expressed in *decimal form*, e.g., $17/10 = 1.7$, $9/100 = 0.09$, $1/6 = 0.16666\dots$. In the case of a rational number the decimal expansion either terminates, or if it does not terminate, one or a group of digits in the expansion will ultimately repeat, as for example, in $\frac{1}{7} = 0.142857142857142\dots$. In the case of an irrational number such as $\sqrt{2} = 1.41423\dots$ or $\pi = 3.14159\dots$ no such repetition can occur. We can always consider a decimal expansion as unending, e.g., 1.375 is the same as 1.37500000 $\dots$ or 1.3749999 $\dots$. To indicate recurring decimals we sometimes place dots over the repeating cycle of digits, e.g., $\frac{1}{7} = 0.\overline{142857}$, $\frac{19}{6} = 3.\overline{16}$.

The decimal system uses the ten digits 0, 1, 2, $\dots$, 9. (These symbols were the gift of the Hindus. They were in use in India by 600 A.D. and then in ensuing centuries were transmitted to the western world by Arab traders.) It is possible to design number systems with fewer or more digits, e.g. the *binary system* uses only two digits 0 and 1 (see Problems 32 and 33).

GEOMETRIC REPRESENTATION OF REAL NUMBERS

The geometric representation of real numbers as points on a line called the *real axis*, as in the figure below, is also well known to the student. For each real number there corresponds one and only one point on the line and conversely, i.e., there is a *one-to-one* (see Fig. 1-1) *correspondence* between the set of real numbers and the set of points on the line. Because of this we often use point and number interchangeably.

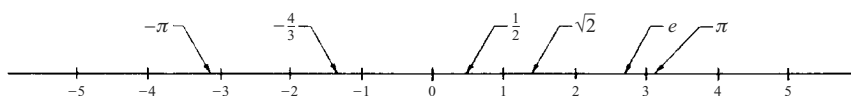


Fig. 1-1

(The interchangeability of point and number is by no means self-evident; in fact, axioms supporting the relation of geometry and numbers are necessary. The Cantor–Dedekind Theorem is fundamental.)

The set of real numbers to the right of 0 is called the set of *positive numbers*; the set to the left of 0 is the set of *negative numbers*, while 0 itself is neither positive nor negative.

(Both the horizontal position of the line and the placement of positive and negative numbers to the right and left, respectively, are conventions.)

Between any two rational numbers (or irrational numbers) on the line there are infinitely many rational (and irrational) numbers. This leads us to call the set of rational (or irrational) numbers an *everywhere dense set*.

OPERATIONS WITH REAL NUMBERS

If a, b, c belong to the set R of real numbers, then:

- | | |
|--|-----------------------------------|
| 1. $a + b$ and ab belong to R | Closure law |
| 2. $a + b = b + a$ | Commutative law of addition |
| 3. $a + (b + c) = (a + b) + c$ | Associative law of addition |
| 4. $ab = ba$ | Commutative law of multiplication |
| 5. $a(bc) = (ab)c$ | Associative law of multiplication |
| 6. $a(b + c) = ab + ac$ | Distributive law |
| 7. $a + 0 = 0 + a = a$, $1 \cdot a = a \cdot 1 = a$ | |

0 is called the *identity with respect to addition*, 1 is called the *identity with respect to multiplication*.

These and extensions to any real numbers are possible so long as division by zero is excluded. In particular, by using 2, with $p = q$ and $p = 0$, respectively, we are lead to the definitions $a^0 = 1$, $a^{-q} = 1/a^q$.

If $a^p = N$, where p is a positive integer, we call a a p th root of N written $\sqrt[p]{N}$. There may be more than one real p th root of N . For example, since $2^2 = 4$ and $(-2)^2 = 4$, there are two real square roots of 4, namely 2 and -2 . For square roots it is customary to define $\sqrt{N}$ as positive, thus $\sqrt{4} = 2$ and then $-\sqrt{4} = -2$.

If p and q are positive integers, we define $a^{p/q} = \sqrt[q]{a^p}$.

LOGARITHMS

If $a^p = N$, p is called the *logarithm* of N to the base a , written $p = \log_a N$. If a and N are positive and $a \neq 1$, there is only one real value for p . The following rules hold:

1. $\log_a MN = \log_a M + \log_a N$
2. $\log_a \frac{M}{N} = \log_a M - \log_a N$
3. $\log_a M^r = r \log_a M$

In practice, two bases are used, base $a = 10$, and the *natural base* $a = e = 2.71828 \dots$. The logarithmic systems associated with these bases are called *common* and *natural*, respectively. The common logarithm system is signified by $\log N$, i.e., the subscript 10 is not used. For natural logarithms the usual notation is $\ln N$.

Common logarithms (base 10) traditionally have been used for computation. Their application replaces multiplication with addition and powers with multiplication. In the age of calculators and computers, this process is outmoded; however, common logarithms remain useful in theory and application. For example, the Richter scale used to measure the intensity of earthquakes is a logarithmic scale. Natural logarithms were introduced to simplify formulas in calculus, and they remain effective for this purpose.

AXIOMATIC FOUNDATIONS OF THE REAL NUMBER SYSTEM

The number system can be built up logically, starting from a basic set of *axioms* or “self-evident” truths, usually taken from experience, such as statements 1–9, Page 2.

If we assume as given the natural numbers and the operations of addition and multiplication (although it is possible to start even further back with the concept of sets), we find that statements 1 through 6, Page 2, with R as the set of natural numbers, hold, while 7 through 9 do not hold.

Taking 7 and 8 as additional requirements, we introduce the numbers $-1, -2, -3, \dots$ and 0. Then by taking 9 we introduce the rational numbers.

Operations with these newly obtained numbers can be defined by adopting axioms 1 through 6, where R is now the set of integers. These lead to *proofs* of statements such as $(-2)(-3) = 6$, $-(-4) = 4$, $(0)(5) = 0$, and so on, which are usually taken for granted in elementary mathematics.

We can also introduce the concept of order or inequality for integers, and from these inequalities for rational numbers. For example, if a, b, c, d are positive integers, we define $a/b > c/d$ if and only if $ad > bc$, with similar extensions to negative integers.

Once we have the set of rational numbers and the rules of inequality concerning them, we can order them geometrically as points on the real axis, as already indicated. We can then show that there are points on the line which do not represent rational numbers (such as $\sqrt{2}$, π , etc.). These irrational numbers can be defined in various ways, one of which uses the idea of *Dedekind cuts* (see Problem 1.34). From this we can show that the usual rules of algebra apply to irrational numbers and that no further real numbers are possible.

POINT SETS, INTERVALS

A set of points (real numbers) located on the real axis is called a *one-dimensional point set*.

The set of points x such that $a \leq x \leq b$ is called a *closed interval* and is denoted by $[a, b]$. The set $a < x < b$ is called an *open interval*, denoted by (a, b) . The sets $a < x \leq b$ and $a \leq x < b$, denoted by $(a, b]$ and $[a, b)$, respectively, are called *half open* or *half closed* intervals.

The symbol x , which can represent any number or point of a set, is called a *variable*. The given numbers a or b are called *constants*.

Letters were introduced to construct algebraic formulas around 1600. Not long thereafter, the philosopher-mathematician Rene Descartes suggested that the letters at the end of the alphabet be used to represent variables and those at the beginning to represent constants. This was such a good idea that it remains the custom.

EXAMPLE. The set of all x such that $|x| < 4$, i.e., $-4 < x < 4$, is represented by $(-4, 4)$, an open interval.

The set $x > a$ can also be represented by $a < x < \infty$. Such a set is called an *infinite* or *unbounded interval*. Similarly, $-\infty < x < \infty$ represents all real numbers x .

COUNTABILITY

A set is called *countable* or *denumerable* if its elements can be placed in 1-1 correspondence with the natural numbers.

EXAMPLE. The even natural numbers $2, 4, 6, 8, \dots$ is a countable set because of the 1-1 correspondence shown.

Given set	2	4	6	8	...
	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	
Natural numbers	1	2	3	4	...

A set is *infinite* if it can be placed in 1-1 correspondence with a subset of itself. An infinite set which is countable is called *countable infinite*.

The set of rational numbers is countable infinite, while the set of irrational numbers or all real numbers is non-countably infinite (see Problems 1.17 through 1.20).

The number of elements in a set is called its *cardinal number*. A set which is countably infinite is assigned the cardinal number $\aleph_0$ (the Hebrew letter *aleph-null*). The set of real numbers (or any sets which can be placed into 1-1 correspondence with this set) is given the cardinal number C , called the *cardinality of the continuum*.

NEIGHBORHOODS

The set of all points x such that $|x - a| < \delta$ where $\delta > 0$, is called a δ *neighborhood* of the point a . The set of all points x such that $0 < |x - a| < \delta$ in which $x = a$ is excluded, is called a *deleted δ neighborhood* of a or an open ball of radius δ about a .

LIMIT POINTS

A *limit point*, *point of accumulation*, or *cluster point* of a set of numbers is a δ number l such that every deleted δ neighborhood of l contains members of the set; that is, no matter how small the radius of a ball about l there are points of the set within it. In other words for any $\delta > 0$, however small, we can always find a member x of the set which is not equal to l but which is such that $|x - l| < \delta$. By considering smaller and smaller values of δ we see that there must be infinitely many such values of x .

A finite set cannot have a limit point. An infinite set may or may not have a limit point. Thus the natural numbers have no limit point while the set of rational numbers has infinitely many limit points.

A set containing all its limit points is called a *closed set*. The set of rational numbers is not a closed set since, for example, the limit point $\sqrt{2}$ is not a member of the set (Problem 1.5). However, the set of all real numbers x such that $0 \leq x \leq 1$ is a closed set.

BOUNDS

If for all numbers x of a set there is a number M such that $x \leq M$, the set is *bounded above* and M is called an *upper bound*. Similarly if $x \geq m$, the set is *bounded below* and m is called a *lower bound*. If for all x we have $m \leq x \leq M$, the set is called *bounded*.

If $\underline{M}$ is a number such that no member of the set is greater than $\underline{M}$ but there is at least one member which exceeds $\underline{M} - \epsilon$ for every $\epsilon > 0$, then $\underline{M}$ is called the *least upper bound* (l.u.b.) of the set. Similarly if no member of the set is smaller than $\bar{m}$ but at least one member is smaller than $\bar{m} + \epsilon$ for every $\epsilon > 0$, then $\bar{m}$ is called the *greatest lower bound* (g.l.b.) of the set.

BOLZANO–WEIERSTRASS THEOREM

The Bolzano–Weierstrass theorem states that every bounded infinite set has at least one limit point. A proof of this is given in Problem 2.23, Chapter 2.

ALGEBRAIC AND TRANSCENDENTAL NUMBERS

A number x which is a solution to the *polynomial equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \quad (I)$$

where $a_0 \neq 0$, $a_1, a_2, \dots, a_n$ are integers and n is a positive integer, called the *degree* of the equation, is called an *algebraic number*. A number which cannot be expressed as a solution of any polynomial equation with integer coefficients is called a *transcendental number*.

EXAMPLES. $\frac{2}{3}$ and $\sqrt{2}$ which are solutions of $3x - 2 = 0$ and $x^2 - 2 = 0$, respectively, are algebraic numbers.

The numbers π and e can be shown to be transcendental numbers. Mathematicians have yet to determine whether some numbers such as $e\pi$ or $e + \pi$ are algebraic or not.

The set of algebraic numbers is a countably infinite set (see Problem 1.23), but the set of transcendental numbers is non-countably infinite.

THE COMPLEX NUMBER SYSTEM

Equations such as $x^2 + 1 = 0$ have no solution within the real number system. Because these equations were found to have a meaningful place in the mathematical structures being built, various mathematicians of the late nineteenth and early twentieth centuries developed an extended system of numbers in which there were solutions. The new system became known as the *complex number system*. It includes the real number system as a subset.

We can consider a complex number as having the form $a + bi$, where a and b are real numbers called the *real* and *imaginary parts*, and $i = \sqrt{-1}$ is called the *imaginary unit*. Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$. We can consider real numbers as a subset of the set of complex numbers with $b = 0$. The complex number $0 + 0i$ corresponds to the real number 0.

The *absolute value* or *modulus* of $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$. The *complex conjugate* of $a + bi$ is defined as $a - bi$. The complex conjugate of the complex number z is often indicated by $\bar{z}$ or z^* .

The set of complex numbers obeys rules 1 through 9 of Page 2, and thus constitutes a field. In performing operations with complex numbers, we can operate as in the algebra of real numbers, replacing i^2 by -1 when it occurs. Inequalities for complex numbers are not defined.

From the point of view of an axiomatic foundation of complex numbers, it is desirable to treat a complex number as an ordered pair (a, b) of real numbers a and b subject to certain operational rules which turn out to be equivalent to those above. For example, we define $(a, b) + (c, d) = (a + c, b + d)$, $(a, b)(c, d) = (ac - bd, ad + bc)$, $m(a, b) = (ma, mb)$, and so on. We then find that $(a, b) = a(1, 0) + b(0, 1)$ and we associate this with $a + bi$, where i is the symbol for $(0, 1)$.

POLAR FORM OF COMPLEX NUMBERS

If real scales are chosen on two mutually perpendicular axes $X'OX$ and $Y'OY$ (the x and y axes) as in Fig. 1-2 below, we can locate any point in the plane determined by these lines by the ordered pair of numbers (x, y) called *rectangular coordinates* of the point. Examples of the location of such points are indicated by P , Q , R , S , and T in Fig. 1-2.

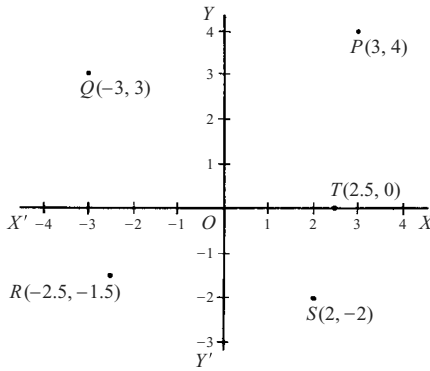


Fig. 1-2

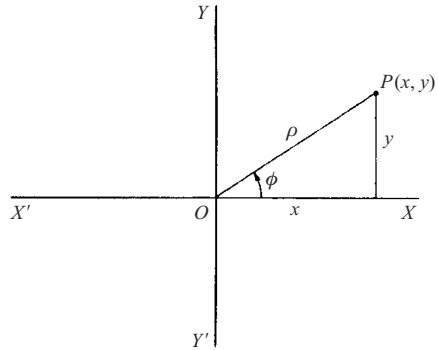


Fig. 1-3

Since a complex number $x + iy$ can be considered as an ordered pair (x, y) , we can represent such numbers by points in an xy plane called the *complex plane* or *Argand diagram*. Referring to Fig. 1-3 above we see that $x = \rho \cos \phi$, $y = \rho \sin \phi$ where $\rho = \sqrt{x^2 + y^2} = |x + iy|$ and ϕ , called the *amplitude* or *argument*, is the angle which line OP makes with the positive x axis OX . It follows that

$$z = x + iy = \rho(\cos \phi + i \sin \phi) \quad (2)$$

called the *polar form* of the complex number, where ρ and ϕ are called *polar coordinates*. It is sometimes convenient to write $\text{cis } \phi$ instead of $\cos \phi + i \sin \phi$.

If $z_1 = x_1 + iy_1 = \rho_1(\cos \phi_1 + i \sin \phi_1)$ and $z_2 = x_2 + iy_2 = \rho_2(\cos \phi_2 + i \sin \phi_2)$ and by using the addition formulas for sine and cosine, we can show that

$$z_1 z_2 = \rho_1 \rho_2 \{\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)\} \quad (3)$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \{\cos(\phi_1 - \phi_2) + i \sin(\phi_1 - \phi_2)\} \quad (4)$$

$$z^n = \{\rho(\cos \phi + i \sin \phi)\}^n = \rho^n (\cos n\phi + i \sin n\phi) \quad (5)$$

where n is any real number. Equation (5) is sometimes called *De Moivre's theorem*. We can use this to determine roots of complex numbers. For example, if n is a positive integer,

$$\begin{aligned} z^{1/n} &= \{\rho(\cos \phi + i \sin \phi)\}^{1/n} \\ &= \rho^{1/n} \left\{ \cos \left(\frac{\phi + 2k\pi}{n} \right) + i \sin \left(\frac{\phi + 2k\pi}{n} \right) \right\} \quad k = 0, 1, 2, 3, \dots, n-1 \end{aligned} \quad (6)$$

from which it follows that there are in general n different values of $z^{1/n}$. Later (Chap. 11) we will show that $e^{i\phi} = \cos \phi + i \sin \phi$ where $e = 2.71828 \dots$. This is called *Euler's formula*.

MATHEMATICAL INDUCTION

The principle of *mathematical induction* is an important property of the positive integers. It is especially useful in proving statements involving all positive integers when it is known for example that the statements are valid for $n = 1, 2, 3$ but it is *suspected* or *conjectured* that they hold for all positive integers. The method of proof consists of the following steps:

1. Prove the statement for $n = 1$ (or some other positive integer).
2. Assume the statement true for $n = k$, where k is any positive integer.
3. From the assumption in 2 prove that the statement must be true for $n = k + 1$. This is part of the proof establishing the induction and may be difficult or impossible.
4. Since the statement is true for $n = 1$ [from step 1] it must [from step 3] be true for $n = 1 + 1 = 2$ and from this for $n = 2 + 1 = 3$, and so on, and so must be true for all positive integers. (This assumption, which provides the link for the truth of a statement for a finite number of cases to the truth of that statement for the infinite set, is called "The Axiom of Mathematical Induction.")

Solved Problems

OPERATIONS WITH NUMBERS

- 1.1.** If $x = 4$, $y = 15$, $z = -3$, $p = \frac{2}{3}$, $q = -\frac{1}{6}$, and $r = \frac{3}{4}$, evaluate (a) $x + (y + z)$, (b) $(x + y) + z$, (c) $p(qr)$, (d) $(pq)r$, (e) $x(p + q)$

(a) $x + (y + z) = 4 + [15 + (-3)] = 4 + 12 = 16$

(b) $(x + y) + z = (4 + 15) + (-3) = 19 - 3 = 16$

The fact that (a) and (b) are equal illustrates the *associative law of addition*.

(c) $p(qr) = \frac{2}{3} \{(-\frac{1}{6})(\frac{3}{4})\} = (\frac{2}{3})(-\frac{3}{24}) = (\frac{2}{3})(-\frac{1}{8}) = -\frac{2}{24} = -\frac{1}{12}$

(d) $(pq)r = \{(\frac{2}{3})(-\frac{1}{6})\}(\frac{3}{4}) = (-\frac{2}{18})(\frac{3}{4}) = (-\frac{1}{9})(\frac{3}{4}) = -\frac{3}{36} = -\frac{1}{12}$

The fact that (c) and (d) are equal illustrates the *associative law of multiplication*.

(e) $x(p + q) = 4(\frac{2}{3} - \frac{1}{6}) = 4(\frac{4}{6} - \frac{1}{6}) = 4(\frac{3}{6}) = \frac{12}{6} = 2$

Another method: $x(p + q) = xp + xq = (4)(\frac{2}{3}) + (4)(-\frac{1}{6}) = \frac{8}{3} - \frac{4}{6} = \frac{8}{3} - \frac{2}{3} = \frac{6}{3} = 2$ using the *distributive law*.

- 1.2.** Explain why we do not consider (a) $\frac{0}{0}$ (b) $\frac{1}{0}$ as numbers.

(a) If we define a/b as that number (if it exists) such that $bx = a$, then $0/0$ is that number x such that $0x = 0$. However, this is true for all numbers. Since there is no unique number which $0/0$ can represent, we consider it undefined.

(b) As in (a), if we define $1/0$ as that number x (if it exists) such that $0x = 1$, we conclude that there is no such number.

Because of these facts we must look upon division by zero as meaningless.

1.3. Simplify $\frac{x^2 - 5x + 6}{x^2 - 2x - 3}$.

$$\frac{x^2 - 5x + 6}{x^2 - 2x - 3} = \frac{(x-3)(x-2)}{(x-3)(x+1)} = \frac{x-2}{x+1} \text{ provided that the cancelled factor } (x-3) \text{ is not zero, i.e., } x \neq 3.$$

For $x = 3$ the given fraction is undefined.

RATIONAL AND IRRATIONAL NUMBERS

1.4. Prove that the square of any odd integer is odd.

Any odd integer has the form $2m + 1$. Since $(2m + 1)^2 = 4m^2 + 4m + 1$ is 1 more than the even integer $4m^2 + 4m = 2(2m^2 + 2m)$, the result follows.

1.5. Prove that there is no rational number whose square is 2.

Let p/q be a rational number whose square is 2, where we assume that p/q is in lowest terms, i.e., p and q have no common integer factors except ± 1 (we sometimes call such integers *relatively prime*).

Then $(p/q)^2 = 2$, $p^2 = 2q^2$ and p^2 is even. From Problem 1.4, p is even since if p were odd, p^2 would be odd. Thus $p = 2m$.

Substituting $p = 2m$ in $p^2 = 2q^2$ yields $q^2 = 2m^2$, so that q^2 is even and q is even.

Thus p and q have the common factor 2, contradicting the original assumption that they had no common factors other than ± 1 . By virtue of this contradiction there can be no rational number whose square is 2.

1.6. Show how to find rational numbers whose squares can be arbitrarily close to 2.

We restrict ourselves to positive rational numbers. Since $(1)^2 = 1$ and $(2)^2 = 4$, we are led to choose rational numbers between 1 and 2, e.g., 1.1, 1.2, 1.3, ..., 1.9.

Since $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$, we consider rational numbers between 1.4 and 1.5, e.g., 1.41, 1.42, ..., 1.49.

Continuing in this manner we can obtain closer and closer rational approximations, e.g. $(1.414213562)^2$ is less than 2 while $(1.414213563)^2$ is greater than 2.

1.7. Given the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_0, a_1, \dots, a_n$ are integers and a_0 and $a_n \neq 0$. Show that if the equation is to have a rational root p/q , then p must divide a_n and q must divide a_0 exactly.

Since p/q is a root we have, on substituting in the given equation and multiplying by q^n , the result

$$a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0 \quad (1)$$

or dividing by p ,

$$a_0p^{n-1} + a_1p^{n-2}q + \cdots + a_{n-1}q^{n-1} = -\frac{a_nq^n}{p} \quad (2)$$

Since the left side of (2) is an integer, the right side must also be an integer. Then since p and q are relatively prime, p does not divide q^n exactly and so must divide a_n .

In a similar manner, by transposing the first term of (1) and dividing by q , we can show that q must divide a_0 .

1.8. Prove that $\sqrt{2} + \sqrt{3}$ cannot be a rational number.

If $x = \sqrt{2} + \sqrt{3}$, then $x^2 = 5 + 2\sqrt{6}$, $x^2 - 5 = 2\sqrt{6}$ and squaring, $x^4 - 10x^2 + 1 = 0$. The only possible rational roots of this equation are ± 1 by Problem 1.7, and these do not satisfy the equation. It follows that $\sqrt{2} + \sqrt{3}$, which satisfies the equation, cannot be a rational number.

1.9. Prove that between any two rational numbers there is another rational number.

The set of rational numbers is closed under the operations of addition and division (non-zero denominator). Therefore, $\frac{a+b}{2}$ is rational. The next step is to guarantee that this value is between a and b . To this purpose, assume $a < b$. (The proof would proceed similarly under the assumption $b < a$.) Then $2a < a + b$, thus $a < \frac{a+b}{2}$ and $a + b < 2b$, therefore $\frac{a+b}{2} < b$.

INEQUALITIES

1.10. For what values of x is $x + 3(2 - x) \geq 4 - x$?

$$x + 3(2 - x) \geq 4 - x \text{ when } x + 6 - 3x \geq 4 - x, 6 - 2x \geq 4 - x, 6 - 4 \geq 2x - x, 2 \geq x, \text{ i.e. } x \leq 2.$$

1.11. For what values of x is $x^2 - 3x - 2 < 10 - 2x$?

The required inequality holds when

$$x^2 - 3x - 2 - 10 + 2x < 0, \quad x^2 - x - 12 < 0 \quad \text{or} \quad (x - 4)(x + 3) < 0$$

This last inequality holds only in the following cases.

Case 1: $x - 4 > 0$ and $x + 3 < 0$, i.e., $x > 4$ and $x < -3$. This is *impossible*, since x cannot be both greater than 4 and less than -3 .

Case 2: $x - 4 < 0$ and $x + 3 > 0$, i.e. $x < 4$ and $x > -3$. This is possible when $-3 < x < 4$. Thus the inequality holds for the set of all x such that $-3 < x < 4$.

1.12. If $a \geq 0$ and $b \geq 0$, prove that $\frac{1}{2}(a + b) \geq \sqrt{ab}$.

The statement is self-evident in the following cases (1) $a = b$, and (2) either or both of a and b zero. For both a and b positive and $a \neq b$, the proof is by contradiction.

Assume to the contrary of the supposition that $\frac{1}{2}(a + b) < \sqrt{ab}$ then $\frac{1}{4}(a^2 + 2ab + b^2) < ab$.

That is, $a^2 - 2ab + b^2 = (a - b)^2 < 0$. Since the left member of this equation is a square, it cannot be less than zero, as is indicated. Having reached this contradiction, we may conclude that our assumption is incorrect and that the original assertion is true.

1.13. If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are any real numbers, prove *Schwarz's inequality*

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$$

For all real numbers λ , we have

$$(a_1\lambda + b_1)^2 + (a_2\lambda + b_2)^2 + \dots + (a_n\lambda + b_n)^2 \geq 0$$

Expanding and collecting terms yields

$$A^2\lambda^2 + 2C\lambda + B^2 \geq 0 \tag{1}$$

where

$$A^2 = a_1^2 + a_2^2 + \dots + a_n^2, \quad B^2 = b_1^2 + b_2^2 + \dots + b_n^2, \quad C = a_1b_1 + a_2b_2 + \dots + a_nb_n \tag{2}$$

The left member of (1) is a quadratic form in λ . Since it never is negative, its discriminant, $4C^2 - 4A^2B^2$, cannot be positive. Thus

$$C^2 - A^2B^2 \leq 0 \quad \text{or} \quad C^2 \leq A^2B^2$$

This is the inequality that was to be proved.

1.14. Prove that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}} < 1$ for all positive integers $n > 1$.

$$\begin{array}{ll}
\text{Let} & S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} \\
\text{Then} & \frac{1}{2} S_n = \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} \\
\text{Subtracting,} & \frac{1}{2} S_n = \frac{1}{2} - \frac{1}{2^n}. \quad \text{Thus } S_n = 1 - \frac{1}{2^{n-1}} < 1 \text{ for all } n.
\end{array}$$

EXPONENTS, ROOTS, AND LOGARITHMS

1.15. Evaluate each of the following:

$$(a) \quad \frac{3^4 \cdot 3^8}{3^{14}} = \frac{3^{4+8}}{3^{14}} = 3^{4+8-14} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$(b) \quad \sqrt{\frac{(5 \cdot 10^{-6})(4 \cdot 10^2)}{8 \cdot 10^5}} = \sqrt{\frac{5 \cdot 4 \cdot 10^{-6} \cdot 10^2}{8 \cdot 10^5}} = \sqrt{2.5 \cdot 10^{-9}} = \sqrt{25 \cdot 10^{-10}} = 5 \cdot 10^{-5} \text{ or } 0.00005$$

$$(c) \quad \log_{2/3}\left(\frac{27}{8}\right) = x. \quad \text{Then } \left(\frac{2}{3}\right)^x = \frac{27}{8} = \left(\frac{3}{2}\right)^3 = \left(\frac{2}{3}\right)^{-3} \text{ or } x = -3$$

$$(d) \quad (\log_a b)(\log_b a) = u. \quad \text{Then } \log_a b = x, \log_b a = y \text{ assuming } a, b > 0 \text{ and } a, b \neq 1.$$

Then $a^x = b$, $b^y = a$ and $u = xy$.

Since $(a^x)^y = a^{xy} = b^y = a$ we have $a^{xy} = a^1$ or $xy = 1$ the required value.

1.16. If $M > 0$, $N > 0$, and $a > 0$ but $a \neq 1$, prove that $\log_a \frac{M}{N} = \log_a M - \log_a N$.

Let $\log_a M = x$, $\log_a N = y$. Then $a^x = M$, $a^y = N$ and so

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y} \quad \text{or} \quad \log_a \frac{M}{N} = x - y = \log_a M - \log_a N$$

COUNTABILITY

1.17. Prove that the set of all rational numbers between 0 and 1 inclusive is countable.

Write all fractions with denominator 2, then 3, ... considering equivalent fractions such as $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \dots$ no more than once. Then the 1-1 correspondence with the natural numbers can be accomplished as follows:

Rational numbers	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	...
	↓	↓	↓	↓	↓	↓	↓	↓	↓	
Natural numbers	1	2	3	4	5	6	7	8	9	...

Thus the set of all rational numbers between 0 and 1 inclusive is countable and has cardinal number $\aleph_0$ (see Page 5).

1.18. If A and B are two countable sets, prove that the set consisting of all elements from A or B (or both) is also countable.

Since A is countable, there is a 1-1 correspondence between elements of A and the natural numbers so that we can denote these elements by $a_1, a_2, a_3, \dots$.

Similarly, we can denote the elements of B by $b_1, b_2, b_3, \dots$.

Case 1: Suppose elements of A are all distinct from elements of B . Then the set consisting of elements from A or B is countable, since we can establish the following 1-1 correspondence.

$A \text{ or } B$	a_1	b_1	a_2	b_2	a_3	b_3	$\dots$
	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	
Natural numbers	1	2	3	4	5	6	$\dots$

Case 2: If some elements of A and B are the same, we count them only once as in Problem 1.17. Then the set of elements belonging to A or B (or both) is countable.

The set consisting of all elements which belong to A or B (or both) is often called the *union* of A and B , denoted by $A \cup B$ or $A + B$.

The set consisting of all elements which are contained in both A and B is called the *intersection* of A and B , denoted by $A \cap B$ or AB . If A and B are countable, so is $A \cap B$.

The set consisting of all elements in A but *not* in B is written $A - B$. If we let $\bar{B}$ be the set of elements which are not in B , we can also write $A - B = A\bar{B}$. If A and B are countable, so is $A - B$.

1.19. Prove that the set of all positive rational numbers is countable.

Consider all rational numbers $x > 1$. With each such rational number we can associate one and only one rational number $1/x$ in $(0, 1)$, i.e., there is a *one-to-one correspondence* between all rational numbers > 1 and all rational numbers in $(0, 1)$. Since these last are countable by Problem 1.17, it follows that the set of all rational numbers > 1 is also countable.

From Problem 1.18 it then follows that the set consisting of all positive rational numbers is countable, since this is composed of the two countable sets of rationals between 0 and 1 and those greater than or equal to 1.

From this we can show that the set of all rational numbers is countable (see Problem 1.59).

1.20. Prove that the set of all real numbers in $[0, 1]$ is non-countable.

Every real number in $[0, 1]$ has a decimal expansion $.a_1a_2a_3\dots$ where $a_1, a_2, \dots$ are any of the digits 0, 1, 2, $\dots$, 9.

We assume that numbers whose decimal expansions terminate such as 0.7324 are written 0.73240000... and that this is the same as 0.73239999...

If all real numbers in $[0, 1]$ are countable we can place them in 1-1 correspondence with the natural numbers as in the following list:

$$\begin{array}{ll} 1 & \leftrightarrow 0.a_{11}a_{12}a_{13}a_{14}\dots \\ 2 & \leftrightarrow 0.a_{21}a_{22}a_{23}a_{24}\dots \\ 3 & \leftrightarrow 0.a_{31}a_{32}a_{33}a_{34}\dots \\ \vdots & \quad \quad \quad \vdots \end{array}$$

We now form a number

$$0.b_1b_2b_3b_4\dots$$

where $b_1 \neq a_{11}, b_2 \neq a_{22}, b_3 \neq a_{33}, b_4 \neq a_{44}, \dots$ and where all b 's beyond some position are not all 9's.

This number, which is in $[0, 1]$ is different from all numbers in the above list and is thus not in the list, contradicting the assumption that all numbers in $[0, 1]$ were included.

Because of this contradiction it follows that the real numbers in $[0, 1]$ cannot be placed in 1-1 correspondence with the natural numbers, i.e., the set of real numbers in $[0, 1]$ is non-countable.

LIMIT POINTS, BOUNDS, BOLZANO-WEIERSTRASS THEOREM

1.21. (a) Prove that the infinite sets of numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ is bounded. (b) Determine the least upper bound (l.u.b.) and greatest lower bound (g.l.b.) of the set. (c) Prove that 0 is a limit point of the set. (d) Is the set a closed set? (e) How does this set illustrate the Bolzano-Weierstrass theorem?

(a) Since all members of the set are less than 2 and greater than -1 (for example), the set is bounded; 2 is an upper bound, -1 is a lower bound.

We can find smaller upper bounds (e.g., $\frac{3}{2}$) and larger lower bounds (e.g., $-\frac{1}{2}$).

- (b) Since no member of the set is greater than 1 and since there is at least one member of the set (namely 1) which exceeds $1 - \epsilon$ for every positive number ϵ , we see that 1 is the l.u.b. of the set.
- Since no member of the set is less than 0 and since there is at least one member of the set which is less than $0 + \epsilon$ for every positive ϵ (we can always choose for this purpose the number $1/n$ where n is a positive integer greater than $1/\epsilon$), we see that 0 is the g.l.b. of the set.
- (c) Let x be any member of the set. Since we can always find a number x such that $0 < |x| < \delta$ for any positive number δ (e.g. we can always pick x to be the number $1/n$ where n is a positive integer greater than $1/\delta$), we see that 0 is a limit point of the set. To put this another way, we see that any deleted δ neighborhood of 0 always includes members of the set, no matter how small we take $\delta > 0$.
- (d) The set is not a closed set since the limit point 0 does not belong to the given set.
- (e) Since the set is bounded and infinite it must, by the Bolzano–Weierstrass theorem, have at least one limit point. We have found this to be the case, so that the theorem is illustrated.

ALGEBRAIC AND TRANSCENDENTAL NUMBERS

1.22. Prove that $\sqrt[3]{2} + \sqrt{3}$ is an algebraic number.

Let $x = \sqrt[3]{2} + \sqrt{3}$. Then $x - \sqrt{3} = \sqrt[3]{2}$. Cubing both sides and simplifying, we find $x^3 + 9x - 2 = 3\sqrt{3}(x^2 + 1)$. Then squaring both sides and simplifying we find $x^6 - 9x^4 - 4x^3 + 27x^2 + 36x - 23 = 0$.

Since this is a polynomial equation with integral coefficients it follows that $\sqrt[3]{2} + \sqrt{3}$, which is a solution, is an algebraic number.

1.23. Prove that the set of all algebraic numbers is a countable set.

Algebraic numbers are solutions to polynomial equations of the form $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ where $a_0, a_1, \dots, a_n$ are integers.

Let $P = |a_0| + |a_1| + \cdots + |a_n| + n$. For any given value of P there are only a finite number of possible polynomial equations and thus only a finite number of possible algebraic numbers.

Write all algebraic numbers corresponding to $P = 1, 2, 3, 4, \dots$ avoiding repetitions. Thus, all algebraic numbers can be placed into 1-1 correspondence with the natural numbers and so are countable.

COMPLEX NUMBERS

1.24. Perform the indicated operations.

$$(a) \quad (4 - 2i) + (-6 + 5i) = 4 - 2i - 6 + 5i = 4 - 6 + (-2 + 5)i = -2 + 3i$$

$$(b) \quad (-7 + 3i) - (2 - 4i) = -7 + 3i - 2 + 4i = -9 + 7i$$

$$(c) \quad (3 - 2i)(1 + 3i) = 3(1 + 3i) - 2i(1 + 3i) = 3 + 9i - 2i - 6i^2 = 3 + 9i - 2i + 6 = 9 + 7i$$

$$(d) \quad \frac{-5 + 5i}{4 - 3i} = \frac{-5 + 5i}{4 - 3i} \cdot \frac{4 + 3i}{4 + 3i} = \frac{(-5 + 5i)(4 + 3i)}{16 - 9i^2} = \frac{-20 - 15i + 20i + 15i^2}{16 + 9}$$

$$= \frac{-35 + 5i}{25} = \frac{5(-7 + i)}{25} = \frac{-7}{5} + \frac{1}{5}i$$

$$(e) \quad \frac{i + i^2 + i^3 + i^4 + i^5}{1 + i} = \frac{i - 1 + (i^2)(i) + (i^2)^2 + (i^2)^2i}{1 + i} = \frac{i - 1 - i + 1 + i}{1 + i}$$

$$= \frac{i}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{i - i^2}{1 - i^2} = \frac{i + 1}{2} = \frac{1}{2} + \frac{1}{2}i$$

$$(f) \quad |3 - 4i||4 + 3i| = \sqrt{(3)^2 + (-4)^2} \sqrt{(4)^2 + (3)^2} = (5)(5) = 25$$

$$(g) \quad \left| \frac{1}{1+3i} - \frac{1}{1-3i} \right| = \left| \frac{1-3i}{1-9i^2} - \frac{1+3i}{1-9i^2} \right| = \left| \frac{-6i}{10} \right| = \sqrt{(0)^2 + \left(-\frac{6}{10}\right)^2} = \frac{3}{5}$$

1.25. If z_1 and z_2 are two complex numbers, prove that $|z_1 z_2| = |z_1| |z_2|$.

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} |z_1 z_2| &= |(x_1 + iy_1)(x_2 + iy_2)| = |x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)| \\ &= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} = \sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2} \\ &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2} \sqrt{x_2^2 + y_2^2} = |x_1 + iy_1| |x_2 + iy_2| = |z_1| |z_2|. \end{aligned}$$

1.26. Solve $x^3 - 2x - 4 = 0$.

The possible rational roots using Problem 1.7 are $\pm 1, \pm 2, \pm 4$. By trial we find $x = 2$ is a root. Then the given equation can be written $(x - 2)(x^2 + 2x + 2) = 0$. The solutions to the quadratic equation

$$ax^2 + bx + c = 0 \quad \text{are} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad \text{For } a = 1, b = 2, c = 2 \text{ this gives } x = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

The set of solutions is $2, -1 + i, -1 - i$.

POLAR FORM OF COMPLEX NUMBERS

1.27. Express in polar form (a) $3 + 3i$, (b) $-1 + \sqrt{3}i$, (c) -1 , (d) $-2 - 2\sqrt{3}i$. See Fig. 1-4.

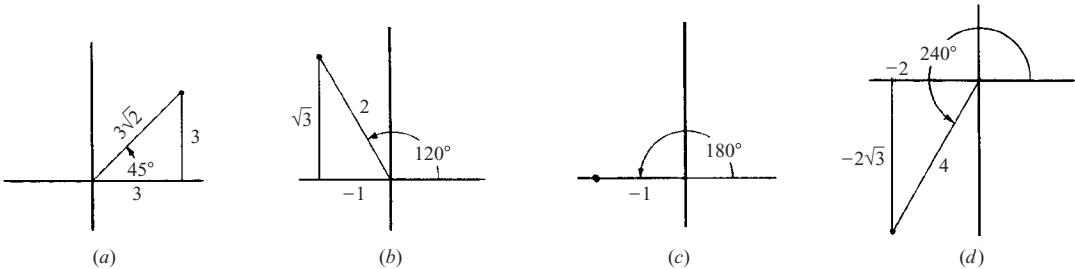


Fig. 1-4

- (a) Amplitude $\phi = 45^\circ = \pi/4$ radians. Modulus $\rho = \sqrt{3^2 + 3^2} = 3\sqrt{2}$. Then $3 + 3i = \rho(\cos \phi + i \sin \phi) = 3\sqrt{2}(\cos \pi/4 + i \sin \pi/4) = 3\sqrt{2} \operatorname{cis} \pi/4 = 3\sqrt{2}e^{i\pi/4}$
- (b) Amplitude $\phi = 120^\circ = 2\pi/3$ radians. Modulus $\rho = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$. Then $-1 + \sqrt{3}i = 2(\cos 2\pi/3 + i \sin 2\pi/3) = 2 \operatorname{cis} 2\pi/3 = 2e^{2\pi i/3}$
- (c) Amplitude $\phi = 180^\circ = \pi$ radians. Modulus $\rho = \sqrt{(-1)^2 + (0)^2} = 1$. Then $-1 = 1(\cos \pi + i \sin \pi) = \operatorname{cis} \pi = e^{i\pi}$
- (d) Amplitude $\phi = 240^\circ = 4\pi/3$ radians. Modulus $\rho = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4$. Then $-2 - 2\sqrt{3}i = 4(\cos 4\pi/3 + i \sin 4\pi/3) = 4 \operatorname{cis} 4\pi/3 = 4e^{4\pi i/3}$

1.28. Evaluate (a) $(-1 + \sqrt{3}i)^{10}$, (b) $(-1 + i)^{1/3}$.

(a) By Problem 1.27(b) and De Moivre's theorem,

$$\begin{aligned} (-1 + \sqrt{3}i)^{10} &= [2(\cos 2\pi/3 + i \sin 2\pi/3)]^{10} = 2^{10}(\cos 20\pi/3 + i \sin 20\pi/3) \\ &= 1024[\cos(2\pi/3 + 6\pi) + i \sin(2\pi/3 + 6\pi)] = 1024(\cos 2\pi/3 + i \sin 2\pi/3) \\ &= 1024(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i) = -512 + 512\sqrt{3}i \end{aligned}$$

(b) $-1 + i = \sqrt{2}(\cos 135^\circ + i \sin 135^\circ) = \sqrt{2}[\cos(135^\circ + k \cdot 360^\circ) + i \sin(135^\circ + k \cdot 360^\circ)]$. Then

$$(-1 + i)^{1/3} = (\sqrt{2})^{1/3} \left[\cos\left(\frac{135^\circ + k \cdot 360^\circ}{3}\right) + i \sin\left(\frac{135^\circ + k \cdot 360^\circ}{3}\right) \right]$$

The results for $k = 0, 1, 2$ are

$$\begin{aligned} &\sqrt[3]{2}(\cos 45^\circ + i \sin 45^\circ), \\ &\sqrt[3]{2}(\cos 165^\circ + i \sin 165^\circ), \\ &\sqrt[3]{2}(\cos 285^\circ + i \sin 285^\circ) \end{aligned}$$

The results for $k = 3, 4, 5, 6, 7, \dots$ give repetitions of these. These complex roots are represented geometrically in the complex plane by points P_1, P_2, P_3 on the circle of Fig. 1-5.

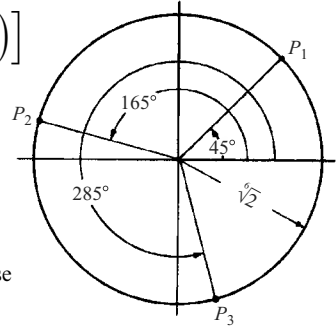


Fig. 1-5

MATHEMATICAL INDUCTION

1.29. Prove that $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

The statement is true for $n = 1$ since $1^2 = \frac{1}{6}(1)(1+1)(2 \cdot 1 + 1) = 1$.

Assume the statement true for $n = k$. Then

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{1}{6}k(k+1)(2k+1)$$

Adding $(k+1)^2$ to both sides,

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 = (k+1)\left[\frac{1}{6}k(2k+1) + k+1\right] \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) = \frac{1}{6}(k+1)(k+2)(2k+3) \end{aligned}$$

which shows that the statement is true for $n = k+1$ if it is true for $n = k$. But since it is true for $n = 1$, it follows that it is true for $n = 1 + 1 = 2$ and for $n = 2 + 1 = 3, \dots$, i.e., it is true for all positive integers n .

1.30. Prove that $x^n - y^n$ has $x - y$ as a factor for all positive integers n .

The statement is true for $n = 1$ since $x^1 - y^1 = x - y$.

Assume the statement true for $n = k$, i.e., assume that $x^k - y^k$ has $x - y$ as a factor. Consider

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k) \end{aligned}$$

The first term on the right has $x - y$ as a factor, and the second term on the right also has $x - y$ as a factor because of the above assumption.

Thus $x^{k+1} - y^{k+1}$ has $x - y$ as a factor if $x^k - y^k$ does.

Then since $x^1 - y^1$ has $x - y$ as factor, it follows that $x^2 - y^2$ has $x - y$ as a factor, $x^3 - y^3$ has $x - y$ as a factor, etc.

- 1.31.** Prove *Bernoulli's inequality* $(1+x)^n > 1+nx$ for $n = 2, 3, \dots$ if $x > -1$, $x \neq 0$.

The statement is true for $n = 2$ since $(1+x)^2 = 1+2x+x^2 > 1+2x$.

Assume the statement true for $n = k$, i.e., $(1+x)^k > 1+kx$.

Multiply both sides by $1+x$ (which is positive since $x > -1$). Then we have

$$(1+x)^{k+1} > (1+x)(1+kx) = 1+(k+1)x+kx^2 > 1+(k+1)x$$

Thus the statement is true for $n = k+1$ if it is true for $n = k$.

But since the statement is true for $n = 2$, it must be true for $n = 2+1 = 3, \dots$ and is thus true for all integers greater than or equal to 2.

Note that the result is not true for $n = 1$. However, the modified result $(1+x)^n \geq 1+nx$ is true for $n = 1, 2, 3, \dots$

MISCELLANEOUS PROBLEMS

- 1.32.** Prove that every positive integer P can be expressed uniquely in the form $P = a_0 2^n + a_1 2^{n-1} + a_2 2^{n-2} + \dots + a_n$ where the a 's are 0's or 1's.

Dividing P by 2, we have $P/2 = a_0 2^{n-1} + a_1 2^{n-2} + \dots + a_{n-1} + a_n/2$.

Then a_n is the remainder, 0 or 1, obtained when P is divided by 2 and is unique.

Let P_1 be the integer part of $P/2$. Then $P_1 = a_0 2^{n-1} + a_1 2^{n-2} + \dots + a_{n-1}$.

Dividing P_1 by 2 we see that a_{n-1} is the remainder, 0 or 1, obtained when P_1 is divided by 2 and is unique.

By continuing in this manner, all the a 's can be determined as 0's or 1's and are unique.

- 1.33.** Express the number 23 in the form of Problem 1.32.

The determination of the coefficients can be arranged as follows:

$$\begin{array}{r} 2)23 \\ 2)11 \quad \text{Remainder 1} \\ 2)5 \quad \text{Remainder 1} \\ 2)2 \quad \text{Remainder 1} \\ 2)1 \quad \text{Remainder 0} \\ 0 \quad \text{Remainder 1} \end{array}$$

The coefficients are 1 0 1 1 1. **Check:** $23 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 1$.

The number 10111 is said to represent 23 in the *scale of two* or *binary scale*.

- 1.34.** Dedekind defined a *cut*, *section*, or *partition* in the rational number system as a separation of *all* rational numbers into two classes or sets called L (the left-hand class) and R (the right-hand class) having the following properties:

- I. The classes are non-empty (i.e. at least one number belongs to each class).
- II. Every rational number is in one class or the other.
- III. Every number in L is less than every number in R .

Prove each of the following statements:

- (a) There cannot be a largest number in L and a smallest number in R .
- (b) It is possible for L to have a largest number and for R to have no smallest number. What type of number does the cut define in this case?
- (c) It is possible for L to have no largest number and for R to have a smallest number. What type of number does the cut define in this case?

- (d) It is possible for L to have no largest number and for R to have no smallest number. What type of number does the cut define in this case?
- (a) Let a be the largest rational number in L , and b the smallest rational number in R . Then either $a = b$ or $a < b$.
 We cannot have $a = b$ since by definition of the cut every number in L is *less* than every number in R .
 We cannot have $a < b$ since by Problem 1.9, $\frac{1}{2}(a + b)$ is a rational number which would be greater than a (and so would have to be in R) but less than b (and so would have to be in L), and by definition a rational number cannot belong to *both* L and R .
- (b) As an indication of the possibility, let L contain the number $\frac{2}{3}$ and all rational numbers less than $\frac{2}{3}$, while R contains all rational numbers greater than $\frac{2}{3}$. In this case the cut defines the rational number $\frac{2}{3}$. A similar argument replacing $\frac{2}{3}$ by any other rational number shows that in such case the cut defines a rational number.
- (c) As an indication of the possibility, let L contain all rational numbers less than $\frac{2}{3}$, while R contains all rational numbers greater than $\frac{2}{3}$. This cut also defines the rational number $\frac{2}{3}$. A similar argument shows that this cut always defines a rational number.
- (d) As an indication of the possibility let L consist of all negative rational numbers and all positive rational numbers whose squares are less than 2, while R consists of all positive numbers whose squares are greater than 2. We can show that if a is any number of the L class, there is always a larger number of the L class, while if b is any number of the R class, there is always a smaller number of the R class (see Problem 1.106). A cut of this type defines an irrational number.
- From (b), (c), (d) it follows that every cut in the rational number system, called a *Dedekind cut*, defines either a rational or an irrational number. By use of Dedekind cuts we can define operations (such as addition, multiplication, etc.) with irrational numbers.

Supplementary Problems

OPERATIONS WITH NUMBERS

- 1.35. Given $x = -3$, $y = 2$, $z = 5$, $a = \frac{3}{2}$, and $b = -\frac{1}{4}$, evaluate:

$$(a) (2x - y)(3y + z)(5x - 2z), \quad (b) \frac{xy - 2z^2}{2ab - 1}, \quad (c) \frac{3a^2b + ab^2}{2a^22b^2 + 1}, \quad (d) \frac{(ax + by)^2 + (ay - bx)^2}{(ay + bx)^2 + (ax - by)^2}.$$

Ans. (a) 2200, (b) 32, (c) $-51/41$, (d) 1

- 1.36. Find the set of values of x for which the following equations are true. Justify all steps in each case.

$$(a) 4\{(x - 2) + 3(2x - 1)\} + 2(2x + 1) = 12(x + 2) - 2 \quad (c) \sqrt{x^2 + 8x + 7} - \sqrt{2x + 2} = x + 1$$

$$(b) \frac{1}{8 - x} - \frac{1}{x - 2} = \frac{1}{4} \quad (d) \frac{1 - x}{\sqrt{x^2 - 2x + 5}} = \frac{3}{5}$$

Ans. (a) 2, (b) 6, -4 (c) $-1, 1$ (d) $-\frac{1}{2}$

- 1.37. Prove that $\frac{x}{(z - x)(x - y)} + \frac{y}{(x - y)(y - z)} + \frac{z}{(y - z)(z - x)} = 0$ giving restrictions if any.

RATIONAL AND IRRATIONAL NUMBERS

- 1.38. Find decimal expansions for (a) $\frac{3}{7}$, (b) $\sqrt{5}$.

Ans. (a) 0.428571 , (b) $2.2360679\dots$

- 1.39.** Show that a fraction with denominator 17 and with numerator $1, 2, 3, \dots, 16$ has 16 digits in the repeating portion of its decimal expansion. Is there any relation between the orders of the digits in these expansions?
- 1.40.** Prove that (a) $\sqrt{3}$, (b) $\sqrt[3]{2}$ are irrational numbers.
- 1.41.** Prove that (a) $\sqrt[3]{5} - \sqrt[4]{3}$, (b) $\sqrt{2} + \sqrt{3} + \sqrt{5}$ are irrational numbers.
- 1.42.** Determine a positive rational number whose square differs from 7 by less than .000001.
- 1.43.** Prove that every rational number can be expressed as a repeating decimal.
- 1.44.** Find the values of x for which
 (a) $2x^3 - 5x^2 - 9x + 18 = 0$, (b) $3x^3 + 4x^2 - 35x + 8 = 0$, (c) $x^4 - 21x^2 + 4 = 0$.
Ans. (a) $3, -2, 3/2$ (b) $8/3, -2 \pm \sqrt{5}$ (c) $\frac{1}{2}(5 \pm \sqrt{17}), \frac{1}{2}(-5 \pm \sqrt{17})$
- 1.45.** If a, b, c, d are rational and m is not a perfect square, prove that $a + b\sqrt{m} = c + d\sqrt{m}$ if and only if $a = c$ and $b = d$.
- 1.46.** Prove that $\frac{1 + \sqrt{3} + \sqrt{5}}{1 - \sqrt{3} + \sqrt{5}} = \frac{12\sqrt{5} - 2\sqrt{15} + 14\sqrt{3} - 7}{11}$.

INEQUALITIES

- 1.47.** Find the set of values of x for which each of the following inequalities holds:
 (a) $\frac{1}{x} + \frac{3}{2x} \geq 5$, (b) $x(x+2) \leq 24$, (c) $|x+2| < |x-5|$, (d) $\frac{x}{x+2} > \frac{x+3}{3x+1}$.
Ans. (a) $0 < x \leq \frac{1}{2}$, (b) $-6 \leq x \leq 4$, (c) $x < 3/2$, (d) $x > 3, -1 < x < -\frac{1}{3}$, or $x < -2$
- 1.48.** Prove (a) $|x+y| \leq |x| + |y|$, (b) $|x+y+z| \leq |x| + |y| + |z|$, (c) $|x-y| \geq |x| - |y|$.
- 1.49.** Prove that for all real x, y, z , $x^2 + y^2 + z^2 \geq xy + yz + zx$.
- 1.50.** If $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$, prove that $ac + bd \leq 1$.
- 1.51.** If $x > 0$, prove that $x^{n+1} + \frac{1}{x^{n+1}} > x^n + \frac{1}{x^n}$ where n is any positive integer.
- 1.52.** Prove that for all real $a \neq 0$, $|a + 1/a| \geq 2$.
- 1.53.** Show that in Schwarz's inequality (Problem 13) the equality holds if and only if $a_p = kb_p$, $p = 1, 2, 3, \dots, n$ where k is any constant.
- 1.54.** If a_1, a_2, a_3 are positive, prove that $\frac{1}{3}(a_1 + a_2 + a_3) \geq \sqrt[3]{a_1 a_2 a_3}$.

EXPONENTS, ROOTS, AND LOGARITHMS

- 1.55.** Evaluate (a) $4^{\log_2 8}$, (b) $\frac{3}{4} \log_{1/8}(\frac{1}{128})$, (c) $\sqrt{\frac{(0.00004)(25,000)}{(0.02)^5(0.125)}}$, (d) $3^{-2 \log_3 5}$, (e) $(-\frac{1}{8})^{4/3} - (-27)^{-2/3}$
Ans. (a) 64, (b) 7/4, (c) 50,000, (d) 1/25, (e) -7/144
- 1.56.** Prove (a) $\log_a MN = \log_a M + \log_a N$, (b) $\log_a M^r = r \log_a M$ indicating restrictions, if any.
- 1.57.** Prove $b^{\log_b a} = a$ giving restrictions, if any.

COUNTABILITY

- 1.58.** (a) Prove that there is a one to one correspondence between the points of the interval $0 \leq x \leq 1$ and $-5 \leq x \leq -3$. (b) What is the cardinal number of the sets in (a)?
Ans. (b) C , the cardinal number of the continuum.
- 1.59.** (a) Prove that the set of all rational numbers is countable. (b) What is the cardinal number of the set in (a)?
Ans. (b) $\aleph_0$
- 1.60.** Prove that the set of (a) all real numbers, (b) all irrational numbers is non-countable.
- 1.61.** The *intersection* of two sets A and B , denoted by $A \cap B$ or AB , is the set consisting of all elements belonging to both A and B . Prove that if A and B are countable, so is their intersection.
- 1.62.** Prove that a countable set of countable sets is countable.
- 1.63.** Prove that the cardinal number of the set of points inside a square is equal to the cardinal number of the sets of points on (a) one side, (b) all four sides. (c) What is the cardinal number in this case? (d) Does a corresponding result hold for a cube?
Ans. (c) C

LIMIT POINTS, BOUNDS, BOLZANO–WEIERSTRASS THEOREM

- 1.64.** Given the set of numbers 1, 1.1, .9, 1.01, .99, 1.001, .999, ... (a) Is the set bounded? (b) Does the set have a l.u.b. and g.l.b.? If so, determine them. (c) Does the set have any limit points? If so, determine them. (d) Is the set a closed set?
Ans. (a) Yes (b) l.u.b. = 1.1, g.l.b. = .9 (c) 1 (d) Yes
- 1.65.** Give the set $-.9, .9, -.99, .99, -.999, .999$ answer the questions of Problem 64.
Ans. (a) Yes (b) l.u.b. = 1, g.l.b. = -1 (c) 1, -1 (d) No
- 1.66.** Give an example of a set which has (a) 3 limit points, (b) no limit points.
- 1.67.** (a) Prove that every point of the interval $0 < x < 1$ is a limit point.
 (b) Are there are limit points which do not belong to the set in (a)? Justify your answer.
- 1.68.** Let S be the set of all rational numbers in $(0, 1)$ having denominator 2^n , $n = 1, 2, 3, \dots$. (a) Does S have any limit points? (b) Is S closed?
- 1.69.** (a) Give an example of a set which has limit points but which is not bounded. (b) Does this contradict the Bolzano–Weierstrass theorem? Explain.

ALGEBRAIC AND TRANSCENDENTAL NUMBERS

- 1.70.** Prove that (a) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, (b) $\sqrt{2}+\sqrt{3}+\sqrt{5}$ are algebraic numbers.
- 1.71.** Prove that the set of transcendental numbers in $(0, 1)$ is not countable.
- 1.72.** Prove that every rational number is algebraic but every irrational number is not necessarily algebraic.

COMPLEX NUMBERS, POLAR FORM

- 1.73.** Perform each of the indicated operations: (a) $2(5-3i) - 3(-2+i) + 5(i-3)$, (b) $(3-2i)^3$
 (c) $\frac{5}{3-4i} + \frac{10}{4+3i}$, (d) $\left(\frac{1-i}{1+i}\right)^{10}$, (e) $\left|\frac{2-4i}{5+7i}\right|^2$, (f) $\frac{(1+i)(2+3i)(4-2i)}{(1+2i)^2(1-i)}$.
Ans. (a) $1-4i$, (b) $-9-46i$, (c) $\frac{11}{5}-\frac{2}{5}i$, (d) -1 , (e) $\frac{10}{37}$, (f) $\frac{16}{5}-\frac{2}{5}i$.

- 1.74.** If z_1 and z_2 are complex numbers, prove (a) $\frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}$, (b) $|z_1^2| = |z_1|^2$ giving any restrictions.
- 1.75.** Prove (a) $|z_1 + z_2| \leq |z_1| + |z_2|$, (b) $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$, (c) $|z_1 - z_2| \geq |z_1| - |z_2|$.
- 1.76.** Find all solutions of $2x^4 - 3x^3 - 7x^2 - 8x + 6 = 0$.
Ans. $3, \frac{1}{2}, -1 \pm i$
- 1.77.** Let z_1 and z_2 be represented by points P_1 and P_2 in the Argand diagram. Construct lines OP_1 and OP_2 , where O is the origin. Show that $z_1 + z_2$ can be represented by the point P_3 , where OP_3 is the diagonal of a parallelogram having sides OP_1 and OP_2 . This is called the *parallelogram law* of addition of complex numbers. Because of this and other properties, complex numbers can be considered as *vectors* in two dimensions.
- 1.78.** Interpret geometrically the inequalities of Problem 1.75.
- 1.79.** Express in polar form (a) $3\sqrt{3} + 3i$, (b) $-2 - 2i$, (c) $1 - \sqrt{3}i$, (d) 5 , (e) $-5i$.
Ans. (a) $6 \operatorname{cis} \pi/6$ (b) $2\sqrt{2} \operatorname{cis} 5\pi/4$ (c) $2 \operatorname{cis} 5\pi/3$ (d) $5 \operatorname{cis} 0$ (e) $5 \operatorname{cis} 3\pi/2$
- 1.80.** Evaluate (a) $[2(\cos 25^\circ + i \sin 25^\circ)][5(\cos 110^\circ + i \sin 110^\circ)]$, (b) $\frac{12 \operatorname{cis} 16^\circ}{(3 \operatorname{cis} 44^\circ)(2 \operatorname{cis} 62^\circ)}$.
Ans. (a) $-5\sqrt{2} + 5\sqrt{2}i$, (b) $-2i$
- 1.81.** Determine all the indicated roots and represent them graphically:
 (a) $(4\sqrt{2} + 4\sqrt{2}i)^{1/3}$, (b) $(-1)^{1/5}$, (c) $(\sqrt{3} - i)^{1/3}$, (d) $i^{1/4}$.
Ans. (a) $2 \operatorname{cis} 15^\circ, 2 \operatorname{cis} 135^\circ, 2 \operatorname{cis} 255^\circ$
 (b) $\operatorname{cis} 36^\circ, \operatorname{cis} 108^\circ, \operatorname{cis} 180^\circ = -1, \operatorname{cis} 252^\circ, \operatorname{cis} 324^\circ$
 (c) $\sqrt[3]{2} \operatorname{cis} 110^\circ, \sqrt[3]{2} \operatorname{cis} 230^\circ, \sqrt[3]{2} \operatorname{cis} 350^\circ$
 (d) $\operatorname{cis} 22.5^\circ, \operatorname{cis} 112.5^\circ, \operatorname{cis} 202.5^\circ, \operatorname{cis} 292.5^\circ$
- 1.82.** Prove that $-1 + \sqrt{3}i$ is an algebraic number.
- 1.83.** If $z_1 = \rho_1 \operatorname{cis} \phi_1$ and $z_2 = \rho_2 \operatorname{cis} \phi_2$, prove (a) $z_1 z_2 = \rho_1 \rho_2 \operatorname{cis}(\phi_1 + \phi_2)$, (b) $z_1/z_2 = (\rho_1/\rho_2) \operatorname{cis}(\phi_1 - \phi_2)$. Interpret geometrically.

MATHEMATICAL INDUCTION

Prove each of the following.

- 1.84.** $1 + 3 + 5 + \cdots + (2n - 1) = n^2$
- 1.85.** $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
- 1.86.** $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{1}{2}n[2a + (n - 1)d]$
- 1.87.** $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$
- 1.88.** $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, r \neq 1$
- 1.89.** $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$
- 1.90.** $1(5) + 2(5)^2 + 3(5)^3 + \cdots + n(5)^{n-1} = \frac{5 + (4n-1)5^{n+1}}{16}$
- 1.91.** $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$ for $n = 1, 2, 3, \dots$

1.92. $(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi$. Can this be proved if n is a rational number?

1.93. $\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}$, $x \neq 0, \pm 2\pi, \pm 4\pi, \dots$

1.94. $\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos \frac{1}{2}x - \cos(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}$, $x \neq 0, \pm 2\pi, \pm 4\pi, \dots$

1.95. $(a + b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \cdots + {}_nC_{n-1}ab^{n-1} + b^n$

where ${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} = {}_nC_{n-r}$. Here $p! = p(p-1)\cdots 1$ and $0!$ is defined as 1. This is called the *binomial theorem*. The coefficients ${}_nC_0 = 1$, ${}_nC_1 = n$, ${}_nC_2 = \frac{n(n-1)}{2!}$, $\dots$, ${}_nC_n = 1$ are called the *binomial coefficients*. ${}_nC_r$ is also written $\binom{n}{r}$.

MISCELLANEOUS PROBLEMS

1.96. Express each of the following integers (scale of 10) in the scale of notation indicated: (a) 87 (two), (b) 64 (three), (c) 1736 (nine). Check each answer.

Ans. (a) 1010111, (b) 2101, (c) 2338

1.97. If a number is 144 in the scale of 5, what is the number in the scale of (a) 2, (b) 8?

1.98. Prove that every rational number p/q between 0 and 1 can be expressed in the form

$$\frac{p}{q} = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} + \cdots$$

where the a 's can be determined uniquely as 0's or 1's and where the process may or may not terminate. The representation $0.a_1a_2\dots a_n\dots$ is then called the *binary form* of the rational number. [Hint: Multiply both sides successively by 2 and consider remainders.]

1.99. Express $\frac{2}{3}$ in the scale of (a) 2, (b) 3, (c) 8, (d) 10.

Ans. (a) 0.1010101..., (b) 0.2 or 0.2000..., (c) 0.5252..., (d) 0.6666...

1.100. A number in the scale of 2 is 11.01001. What is the number in the scale of 10.

Ans. 3.28125

1.101. In what scale of notation is $3 + 4 = 12$?

Ans. 5

1.102. In the scale of 12, two additional symbols t and e must be used to designate the "digits" 10 and 11, respectively. Using these symbols, represent the integer 5110 (scale of 10) in the scale of 12.

Ans. $2e5t$

1.103. Find a rational number whose decimal expansion is 1.636363....

Ans. $18/11$

1.104. A number in the scale of 10 consists of six digits. If the last digit is removed and placed before the first digit, the new number is one-third as large. Find the original number.

Ans. 428571

1.105. Show that the rational numbers form a field.

1.106. Using as axioms the relations 1–9 on Pages 2 and 3, prove that

(a) $(-3)(0) = 0$, (b) $(-2)(+3) = -6$, (c) $(-2)(-3) = 6$.

- 1.107.** (a) If x is a rational number whose square is less than 2, show that $x + (2 - x^2)/10$ is a larger such number.
(b) If x is a rational number whose square is greater than 2, find in terms of x a smaller rational number whose square is greater than 2.
- 1.108.** Illustrate how you would use Dedekind cuts to define
(a) $\sqrt{5} + \sqrt{3}$, (b) $\sqrt{3} - \sqrt{2}$, (c) $(\sqrt{3})(\sqrt{2})$, (d) $\sqrt{2}/\sqrt{3}$.

CHAPTER 2

Sequences

DEFINITION OF A SEQUENCE

A sequence is a set of numbers $u_1, u_2, u_3, \dots$ in a definite order of arrangement (i.e., a *correspondence* with the natural numbers) and formed according to a definite rule. Each number in the sequence is called a *term*; u_n is called the *n*th *term*. The sequence is called *finite* or *infinite* according as there are or are not a finite number of terms. The sequence $u_1, u_2, u_3, \dots$ is also designated briefly by $\{u_n\}$.

EXAMPLES. 1. The set of numbers $2, 7, 12, 17, \dots, 32$ is a finite sequence; the *n*th term is given by $u_n = 2 + 5(n - 1) = 5n - 3$, $n = 1, 2, \dots, 7$.
2. The set of numbers $1, 1/3, 1/5, 1/7, \dots$ is an infinite sequence with *n*th term $u_n = 1/(2n - 1)$, $n = 1, 2, 3, \dots$

Unless otherwise specified, we shall consider infinite sequences only.

LIMIT OF A SEQUENCE

A number l is called the *limit* of an infinite sequence $u_1, u_2, u_3, \dots$ if for any positive number ϵ we can find a positive number N depending on ϵ such that $|u_n - l| < \epsilon$ for all integers $n > N$. In such case we write $\lim_{n \rightarrow \infty} u_n = l$.

EXAMPLE . If $u_n = 3 + 1/n = (3n + 1)/n$, the sequence is $4, 7/2, 10/3, \dots$ and we can show that $\lim_{n \rightarrow \infty} u_n = 3$.

If the limit of a sequence exists, the sequence is called *convergent*; otherwise, it is called *divergent*. A sequence can converge to only one limit, i.e., if a limit exists, it is unique. See Problem 2.8.

A more intuitive but unrigorous way of expressing this concept of limit is to say that a sequence $u_1, u_2, u_3, \dots$ has a limit l if the successive terms get “closer and closer” to l . This is often used to provide a “guess” as to the value of the limit, after which the definition is applied to see if the guess is really correct.

THEOREMS ON LIMITS OF SEQUENCES

If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$, then

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = A + B$
2. $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n = A - B$
3. $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = (\lim_{n \rightarrow \infty} a_n)(\lim_{n \rightarrow \infty} b_n) = AB$

4. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{A}{B}$ if $\lim_{n \rightarrow \infty} b_n = B \neq 0$
 If $B = 0$ and $A \neq 0$, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ does not exist.
 If $B = 0$ and $A = 0$, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ may or may not exist.
5. $\lim_{n \rightarrow \infty} a_n^p = (\lim_{n \rightarrow \infty} a_n)^p = A^p$, for $p =$ any real number if A^p exists.
6. $\lim_{n \rightarrow \infty} p^{a_n} = p^{\lim_{n \rightarrow \infty} a_n} = p^A$, for $p =$ any real number if p^A exists.

INFINITY

We write $\lim_{n \rightarrow \infty} a_n = \infty$ if for each positive number M we can find a positive number N (depending on M) such that $a_n > M$ for all $n > N$. Similarly, we write $\lim_{n \rightarrow \infty} a_n = -\infty$ if for each positive number M we can find a positive number N such that $a_n < -M$ for all $n > N$. It should be emphasized that ∞ and $-\infty$ are not numbers and the sequences are not convergent. The terminology employed merely indicates that the sequences diverge in a certain manner. That is, no matter how large a number in absolute value that one chooses there is an n such that the absolute value of a_n is greater than that quantity.

BOUNDED, MONOTONIC SEQUENCES

If $u_n \leq M$ for $n = 1, 2, 3, \dots$, where M is a constant (independent of n), we say that the sequence $\{u_n\}$ is *bounded above* and M is called an *upper bound*. If $u_n \geq m$, the sequence is *bounded below* and m is called a *lower bound*.

If $m \leq u_n \leq M$ the sequence is called *bounded*. Often this is indicated by $|u_n| \leq P$. Every convergent sequence is bounded, but the converse is not necessarily true.

If $u_{n+1} \geq u_n$ the sequence is called *monotonic increasing*; if $u_{n+1} > u_n$ it is called *strictly increasing*.

Similarly, if $u_{n+1} \leq u_n$ the sequence is called *monotonic decreasing*, while if $u_{n+1} < u_n$ it is *strictly decreasing*.

- EXAMPLES.**
1. The sequence 1, 1.1, 1.11, 1.111, ... is bounded and monotonic increasing. It is also strictly increasing.
 2. The sequence 1, -1, 1, -1, 1, ... is bounded but not monotonic increasing or decreasing.
 3. The sequence -1, -1.5, -2, -2.5, -3, ... is monotonic decreasing and not bounded. However, it is bounded above.

The following theorem is fundamental and is related to the Bolzano–Weierstrass theorem (Chapter 1, Page 6) which is proved in Problem 2.23.

Theorem. Every bounded monotonic (increasing or decreasing) sequence has a limit.

LEAST UPPER BOUND AND GREATEST LOWER BOUND OF A SEQUENCE

A number $\underline{M}$ is called the *least upper bound* (l.u.b.) of the sequence $\{u_n\}$ if $u_n \leq \underline{M}$, $n = 1, 2, 3, \dots$ while at least one term is greater than $\underline{M} - \epsilon$ for any $\epsilon > 0$.

A number $\bar{m}$ is called the *greatest lower bound* (g.l.b.) of the sequence $\{u_n\}$ if $u_n \geq \bar{m}$, $n = 1, 2, 3, \dots$ while at least one term is less than $\bar{m} + \epsilon$ for any $\epsilon > 0$.

Compare with the definition of l.u.b. and g.l.b. for sets of numbers in general (see Page 6).

LIMIT SUPERIOR, LIMIT INFERIOR

A number $\bar{l}$ is called the *limit superior, greatest limit or upper limit* ($\limsup$ or $\overline{\lim}$) of the sequence $\{u_n\}$ if infinitely many terms of the sequence are greater than $\bar{l} - \epsilon$ while only a finite number of terms are greater than $\bar{l} + \epsilon$, where ϵ is any positive number.

A number $\underline{l}$ is called the *limit inferior, least limit or lower limit* ($\liminf$ or $\underline{\lim}$) of the sequence $\{u_n\}$ if infinitely many terms of the sequence are less than $\underline{l} + \epsilon$ while only a finite number of terms are less than $\underline{l} - \epsilon$, where ϵ is any positive number.

These correspond to least and greatest limiting points of general sets of numbers.

If infinitely many terms of $\{u_n\}$ exceed any positive number M , we define $\limsup \{u_n\} = \infty$. If infinitely many terms are less than $-M$, where M is any positive number, we define $\liminf \{u_n\} = -\infty$.

If $\lim_{n \rightarrow \infty} u_n = \infty$, we define $\limsup \{u_n\} = \liminf \{u_n\} = \infty$.

If $\lim_{n \rightarrow \infty} u_n = -\infty$, we define $\limsup \{u_n\} = \liminf \{u_n\} = -\infty$.

Although every bounded sequence is not necessarily convergent, it always has a finite $\limsup$ and $\liminf$.

A sequence $\{u_n\}$ converges if and only if $\limsup u_n = \liminf u_n$ is finite.

NESTED INTERVALS

Consider a set of intervals $[a_n, b_n]$, $n = 1, 2, 3, \dots$, where each interval is contained in the preceding one and $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$. Such intervals are called *nested intervals*.

We can prove that to every set of nested intervals there corresponds one and only one real number. This can be used to establish the Bolzano–Weierstrass theorem of Chapter 1. (See Problems 2.22 and 2.23.)

CAUCHY'S CONVERGENCE CRITERION

Cauchy's convergence criterion states that a sequence $\{u_n\}$ converges if and only if for each $\epsilon > 0$ we can find a number N such that $|u_p - u_q| < \epsilon$ for all $p, q > N$. This criterion has the advantage that one need not know the limit l in order to demonstrate convergence.

INFINITE SERIES

Let $u_1, u_2, u_3, \dots$ be a given sequence. Form a new sequence $S_1, S_2, S_3, \dots$ where

$$S_1 = u_1, S_2 = u_1 + u_2, S_3 = u_1 + u_2 + u_3, \dots, S_n = u_1 + u_2 + u_3 + \dots + u_n, \dots$$

where S_n , called the *n th partial sum*, is the sum of the first n terms of the sequence $\{u_n\}$.

The sequence $S_1, S_2, S_3, \dots$ is symbolized by

$$u_1 + u_2 + u_3 + \dots = \sum_{n=1}^{\infty} u_n$$

which is called an *infinite series*. If $\lim_{n \rightarrow \infty} S_n = S$ exists, the series is called *convergent* and S is its *sum*, otherwise the series is called *divergent*.

Further discussion of infinite series and other topics related to sequences is given in Chapter 11.

Solved Problems

SEQUENCES

2.1. Write the first five terms of each of the following sequences.

$$(a) \left\{ \frac{2n-1}{3n+2} \right\}$$

$$(b) \left\{ \frac{1 - (-1)^n}{n^3} \right\}$$

$$(c) \left\{ \frac{(-1)^{n-1}}{2 \cdot 4 \cdot 6 \cdots 2n} \right\}$$

$$(d) \left\{ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \right\}$$

$$(e) \left\{ \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \right\}$$

$$(a) \frac{1}{5}, \frac{3}{8}, \frac{5}{11}, \frac{7}{14}, \frac{9}{17}$$

$$(b) \frac{2}{1^3}, 0, \frac{2}{3^3}, 0, \frac{2}{5^3}$$

$$(c) \frac{1}{2}, \frac{-1}{2 \cdot 4}, \frac{1}{2 \cdot 4 \cdot 6}, \frac{-1}{2 \cdot 4 \cdot 6 \cdot 8}, \frac{1}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}$$

$$(d) \frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

$$(e) \frac{x}{1!}, \frac{-x^3}{3!}, \frac{x^5}{5!}, \frac{-x^7}{7!}, \frac{x^9}{9!}$$

Note that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$. Thus $1! = 1$, $3! = 1 \cdot 2 \cdot 3 = 6$, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$, etc. We define $0! = 1$.

2.2. Two students were asked to write an n th term for the sequence 1, 16, 81, 256, ... and to write the 5th term of the sequence. One student gave the n th term as $u_n = n^4$. The other student, who did not recognize this simple law of formation, wrote $u_n = 10n^3 - 35n^2 + 50n - 24$. Which student gave the correct 5th term?

If $u_n = n^4$, then $u_1 = 1^4 = 1$, $u_2 = 2^4 = 16$, $u_3 = 3^4 = 81$, $u_4 = 4^4 = 256$, which agrees with the first four terms of the sequence. Hence the first student gave the 5th term as $u_5 = 5^4 = 625$.

If $u_n = 10n^3 - 35n^2 + 50n - 24$, then $u_1 = 1$, $u_2 = 16$, $u_3 = 81$, $u_4 = 256$, which also agrees with the first four terms given. Hence, the second student gave the 5th term as $u_5 = 601$.

Both students were correct. Merely giving a finite number of terms of a sequence does not define a unique n th term. In fact, an infinite number of n th terms is possible.